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COMMON FIXED POINT THEOREMS FOR GENERALIZED CONTRACTION MAPPINGS ON UNIFORM SPACES

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Abstract

In this paper, we establish common fixed point theorems for weakly compatible mappings in uniform spaces by employing the concepts of *A*-distance and *E*-distance as well as comparison functions. Our results extend and generalize some results in the literature.

1. Introduction

Fixed point theorems for contraction mappings in uniform spaces have gained certain scope in analysis. The introduction of *E*-distance and *A*-distance in uniform spaces by Aamri and El Moutawakil [1] has motivated many researchers to investigate more on fixed point theorems in uniform spaces. Jungck and Rhoades [2] initiated the existence of common fixed point for weakly compatible mappings on metric spaces. Thereafter, many authors employed the ideas in proving several common fixed point theorems

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in different abstract spaces (see [3-5]). In this paper, we prove the common fixed point theorems for weakly compatible mappings employing the notion of *A*-distance and *E*-distance in uniform spaces.

2. Preliminaries

The following definitions and motivations are needed in the sequel.

Definition 2.1 [2]**.** Two selfmaps *f* and *g* of a metric space *X* are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $fu = gu$ for $u \in X$, then $fgu = gfu$.

Definition 2.2 [6]**.** Let *X* be a nonempty set and *U* and *V* be nonempty subsets of $X \times Y$. Suppose that

$\Delta(x, x) = \{(x, x) : x \in X\}$ is the diagonal of *X*;

$U^{-1} = \{(x, y) : (y, x) \in U\}$ is the inverse of *U*;

$UoV = \{(x, y) : \exists x \in X \text{ s.t. } (x, z) \in V, (z, y) \in U\}$.

Then (X, ϕ) is a *uniform space* if a nonempty set *X* equipped with a nonempty family ϕ of subsets of $X \times X$ satisfies the following properties:

(i) Every $U \in \phi$ contains the diagonal $\{(x, x) : x \in \mathbb{R}^+\}$.

(ii) If $U \in \phi$, so does U^{-1}.

(iii) If $U \in \phi$, then there exists $V \in \phi$ such that, whenever (x, y) and (y, z) are in *V*, then (x, z) is in *U*.

Definition 2.3 [1]**.** Let (X, ϕ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an *A-distance* if for any $V \in \phi$ there exists $\delta > 0$ such that, if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2.4 [1]**.** Let (X, ϕ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an *E-distance* if *p* is an *A*-distance and $p(x, y) \leq p(x, z) + p(z, y)$, for every $x, y, z \in X$.

Definition 2.5 [1]**.** Let (X, ϕ) be a uniform space and p be an A-distance on X. Then

(i) X is *S-complete* if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that $\lim p(x_n, x) = 0$.

(ii) X is *p-Cauchy complete* if every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that $\lim x_n = x$ with respect to $\tau(\phi)$.

(iii) $f : X \to X$ is *p-continuous* if $\lim p(x_n, x) = 0$ implies $\lim p(fx_n, fx) = 0$.

(iv) $f : X \to X$ is $\tau(\phi)$*-continuous* if $\lim x_n = x$ with respect to $\tau(\phi)$ implies $\lim fx_n = fx$ with respect to $\tau(\phi)$.

(v) X is said to be *p-bounded* if $\delta(X) = \sup\{p(x, y) : x, y \in X\} < \infty$.

Lemma 2.6 [3]**.** *Let (X, ϕ) be a uniform space and p be an A-distance on X. Let $\{x_n\}, \{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to* 0. *Then for $x, y, z \in X$, the following hold*:

(a) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$*

(b) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*

(c) *If $p(x_n, x_m) \leq \alpha_n$ for all $n > m$, then $\{x_n\}$ is a Cauchy sequence in* (X, ϕ).

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and satisfying the conditions:

(i) ψ is nondecreasing on $\mathbb{R}^+$,

(ii) $0 < \psi(t) < t$, for each $t \in (0, \infty)$.

3. Main Results

Theorem 3.1. *Let* (X, ϕ) *be a Hausdorff uniform space and p be an A-distance on X. Let f and g be weakly compatible mappings satisfying the following conditions*:

(i) $f(X) \subseteq g(X)$,

(ii)

$$p(fx, fy) \leq \psi\left(\max\left\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{p(gx, fy) + p(gy, fx)}{2}\right\}\right). \quad (3.1)$$

If $f(X)$ *or* $g(X)$ *is an S-complete subspace of X, then f and g have a common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. Choose $x_2 \in X$ such that $f(x_1) = g(x_2)$. In general, we can construct $\{x_n\} \in X$ such that $f(x_n) = g(x_{n+1})$. Let the sequence y_{n+1} be defined as $y_{n+1} = f(x_n) = g(x_{n+1})$.

Since $g(X)$ is *S*-complete, there is $z \in X$ such that $z = fx_n = gx_n = fx_{n+1} = gx_{n+1} = y_{n+1}$. Taking $u = x_n$ yields $f(u) = g(u)$. With the weakly compatibility of *f* and *g*, we have $fz = fgu = gfu = gz$. Now we have the following cases to deal with:

Case (1)

$$\begin{aligned} p(fz, z) = p(fz, fu) &\leq \psi\left(\max\left\{p(gz, gu), p(gz, fz), p(gu, fu), \frac{p(gz, fu) + p(gu, fz)}{2}\right\}\right) \\ &\leq \psi\left(\max\left\{p(z, fz), p(z, fu), \frac{p(z, z) + p(z, fz)}{2}\right\}\right) \\ &\leq \psi\left(\max\left\{p(z, fz), p(z, z), \frac{p(z, z) + p(z, fz)}{2}\right\}\right) \end{aligned}$$

$$\leq \psi\left(\max\left\{p(z, fz), \frac{p(z, fz)}{2}\right\}\right)$$

$$\leq \psi(p(z, fz)) < p(z, fz).$$

It is a contradiction, hence $z = fz$. This means that z is the common fixed point of f and g.

Case (2)

If $y_n \neq y_{n+p}$ for all $n \in \mathbb{N}$ and since $g(X)$ is S-complete, then there is $z \in X$ such that $z = \lim y_n = \lim fx_n = \lim gx_{n+1}$.

With $z \in g(X)$, there exists $w \in X$ such that $z = gw$. Using (3.1), we obtain

$$p(fw, gw) \leq p(fw, fx_n) + p(fx_n, gw)$$

$$\leq \psi\left(\max\left\{p(gw, gx_n), p(gw, fw), p(gx_n, fx_n),\right.\right.$$

$$\left.\left.\frac{p(gw, fx_n) + p(gx_n, fw)}{2}\right\}\right) + p(z, z)$$

$$\leq \psi\left(\max\left\{p(z, z), p(gw, fw), p(z, z), \frac{p(z, z) + p(gw, fw)}{2}\right\}\right)$$

$$\leq \psi\left(\max\left\{p(gw, fw), \frac{p(gw, fw)}{2}\right\}\right)$$

$$\leq \psi(p(gw, fw)) < p(gw, fw).$$

It is a contradiction, hence $fw = gw$. Since f and g are weakly compatible, we obtain $gz = gfw = fgw = fz$. Next, we prove that $z = Fz$. On the contrary, let $z \neq fz$, then using (3.1) we get

$$p(z, fz)$$

$$= p(fw, fz)$$

$$\leq \psi\left(\max\left\{p(gz, gw), p(gz, fz), p(gw, fw), \frac{p(gz, fw) + p(gw, fz)}{2}\right\}\right)$$

$$\leq \psi\left(\max\left\{p(z, z), p(z, fz), 0, \frac{p(z, z) + p(z, fz)}{2}\right\}\right)$$

$$\leq \psi\left(\max\left\{p(z, fz), \frac{p(z, fz)}{2}\right\}\right)$$

$$\leq \psi(p(z, fz)) < p(z, fz).$$

It is a contradiction, hence $p(z, fz) = 0.$ Following the same procedure, we have that $p(fz, z) = 0.$ By the definition of uniform space, we obtain

$$p(z, z) \leq p(z, fz) + p(fz, z) = 0.$$

Since $p(z, z) \geq 0,$ we get $p(z, z) = 0.$ Using Lemma 2.6, we have $fz = z.$ Therefore, z is the common fixed point of f and g.

To prove the uniqueness of the common fixed point of f and g, we define E-distance on uniform spaces. This leads to the following result.

Theorem 3.2. *Let (X, ϕ) be a Hausdorff uniform space and p be an E-distance on X. Let f and g be weakly compatible mappings satisfying the following conditions*:

(i) $f(X) \subseteq g(X),$

(ii) $p(fx, fy)$

$$\leq \psi\left(\max\left\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{p(gx, fy) + p(gy, fx)}{2}\right\}\right). \tag{3.2}$$

If $f(X)$ or $g(X)$ is an S-complete subspace of X, then f and g have a unique common fixed point.

Proof. It has been established in Theorem 3.1 that f and g have a common fixed point. Now we prove that f and g have unique common fixed point. Suppose f and g have two different common fixed points (say) z_1 and

z_2. Then we show that $z_1 = z_2$. On the contrary, let $z_1 \neq z_2$. Then, using (3.2), we obtain

$$p(z_1, z_2)$$

$$= p(fz_1, fz_2)$$

$$\leq \psi\left(\max\left\{p(gz_1, gz_2), p(gz_1, fz_1), p(gz_2, fz_2), \frac{p(gz_1, fz_2) + p(gz_2, fz_1)}{2}\right\}\right)$$

$$\leq \psi\left(\max\left\{p(z_1, z_1), p(z_1, z_1), p(z_2, z_2), \frac{p(z_1, z_2) + p(z_2, z_1)}{2}\right\}\right)$$

$$\leq \psi(\max\{p(z_1, z_2), p(z_1, z_2)\})$$

$$\leq \psi(p(z_1, z_2)) < p(z_1, z_2).$$

It is a contradiction, hence $p(z_1, z_2) = 0$. Similarly, we can obtain $p(z_2, z_1) = 0$. By the definition of uniform spaces, we have $p(z_1, z_1) \leq p(z_1, z_2) + p(z_2, z_1) = 0$.

Therefore, applying Lemma 2.6 yields $z_1 = z_2$. Thus, z is the unique common fixed point of f and g.

Corollary 3.3. *Let (X, ϕ) be a Hausdorff uniform space and p be an A-distance on X. Let f and g be weakly compatible mappings satisfying the following conditions*:

(i) $f(X) \subseteq g(X)$,

(ii) $p(fx, fy) \leq \psi(p(gx, gy))$. (3.3)

If $f(X)$ or $g(X)$ is an S-complete subspace of X, then f and g have a common fixed point.

Remarks 3.4. Theorem 3.1 and Theorem 3.2 are generalizations of ([1], Theorem 3.1 and Theorem 3.3) with respect to their maps. It is also an extension of the result of ([3], Theorem 1 and Theorem 2).

Example 3.5. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Suppose f and g

are defined by $f(x) = \frac{x^2}{4}$ if $x \in \left[0, \frac{1}{4}\right)$ and $f(x) = 1$ if $x \in \left[\frac{1}{4}, 1\right]$. Also, $g(x) = x^2$ if $x \in \left[0, \frac{1}{4}\right)$ and $g(x) = 0$ if $x \in \left[\frac{1}{4}, 1\right]$. Let functions p and ψ be defined as follows: if $\psi(x) = \frac{x}{4}$ and $p(x, y) = 0$ if $y \in \left[0, \frac{1}{4}\right)$ and $p(x, y) = y$ if $y \in \left[\frac{1}{4}, 1\right]$. All the conditions of Theorem 3.2 are satisfied and the common fixed point of f and g is $\frac{3}{4}$.

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