CONTINUOUS IMPLICIT HYBRID ONE-STEP METHODS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS OF GENERAL SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

BY

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A Ph.D. Thesis

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Declaration

I, Anake, Timothy Ashibel, (Matric. Number: CUGP060192) declare that this research was carried out by me under the supervision of Prof. David O. Awoyemi of the Department of Mathematical Sciences, Federal University of Technology, Akure and Dr. Johnson O. Olaleru of the Department of Mathematics, University of Lagos, Lagos. I attest that the thesis has not been presented either wholly or partly for the award of any degree elsewhere. All sources of data and schorlarly information used in this thesis are duly acknowledged.

Signature...............................

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Certification

This is to certify that this Research was carried out by **Anake, Timothy Ashibel, (CUGP060192)**, in the Department of Mathematics, School of Natural and Applied Sciences, College of Science and Technology, Covenant University, Ota, Nigeria.

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Dedication

To my lovely wife, Winifred and the memory of my late brother, Chris (a.k.a cobra).

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Abstract

The numerical solutions of initial value problems of general second order ordinary differential equations have been studied in this work. A new class of continuous implicit hybrid one step methods capable of solving initial value problems of general second order ordinary differential equations has been developed using the collocation and interpolation technique on the power series approximate solution. The one step method was augmented by the introduction of offstep points in order to circumvent Dahlquist zero stability barrier and upgrade the order of consistency of the methods. The new class of continuous implicit hybrid one step methods has the advantage of easy change of step length and evaluation of functions at offstep points. The Block method used to implement the main method guarantees that each discrete method obtained from the simultaneous solution of the block has the same order of accuracy as the main method. Hence, the new class of one step methods gives high order of accuracy with very low error constants, gives large intervals of absolute stability, are zero stable and converge. Sample examples of linear, nonlinear and stiff problems have been used to test the performance of the methods as well as to compare computed results and the associated errors with the exact solutions and errors of results obtained from existing methods, respectively, in terms of step number and order of accuracy, using written efficient computer codes.

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Chapter 1

Introduction

1.1 Preambles

In science and engineering, usually mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations. Differential equations do not only arise in the physical sciences but also in diverse fields as economics, medicine, psychology, operation research and even in areas such as biology and anthropology.

Interestingly, differential equations arising from the modeling of physical phenomena, often do not have analytic solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary. To that extent, several numerical methods such as finite difference methods, finite element methods and finite volume methods, among others, have been developed based on the nature and type of the differential equation to be solved.

A differential equation in which the unknown function is a function of two or more independent variable is called partial differential equation. Those in which the unknown function is a function of only one independent variable are called ordinary differential equations. This work concerns the study of numerical solutions of the latter.

In particular, finite difference methods have excelled for the numerical treatment of ordinary differential equations especially since the advent of digital computers. The development of algorithms has been largely guided by convergence theorems of Dahlquist (1956, 1959, 1963, 1978) as well as the treatises of Henrici (1962) and Stetter (1973), (Fatunla, 1988).

The development of numerical methods for the solution of Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form

$$y^{(\mu)} = f(x, y, y^{(1)}, \ldots, y^{(\mu-1)}), y(a) = \eta_0,\ y^{(1)}(a) = \eta_1, \ldots, y^{(\mu-1)}(a) = \eta_{\mu-1} \quad (1.1.1)$$

on the interval $[a, b]$ has given rise to two major discrete variable methods namely; one step (or single step) methods and multistep methods especially the Linear Multistep Methods (LMMs).

One step methods include the Euler's methods, the Runge-Kutta methods, the theta methods, etc. These methods are only suitable for the solutions of first order IVPs of ODEs because of their very low order of accuracy. In order to develop higher order one step methods such as Runge-Kutta methods, the efficiency of Euler methods, in terms of the number of function evaluations per step, is sacrificed since more function evaluations is required. Hence, solving (1.1.1) using any one step method means reducing it to an equivalent system of first order IVPs of ODEs which increases the dimension of the problem thus increasing its scale. The result is that one step methods become time-consuming for large scale problems and give results of low accuracy.

Linear multistep methods on the other hand, include methods such as Adam-Bashforth method, Adam-Moulton method, and Numerov method. These methods give high order of accuracy and are suitable for the direct solution of (1.1.1) without necessarily reducing it to an equivalent system of first order IVPs of ODEs. Linear multistep methods are not as efficient, in terms of function evaluations, as the one step method and also require some values to start the integration process.

This research work is concerned with the development of continuous implicit hybrid one step methods. These methods combine the efficiency of one step methods and the high order of accuracy of multistep methods to solve the particular case of (1.1.1) when $n = 2$.

Basically, the thesis consists of six chapters. Chapter one contains the introduction, basic features of ordinary differential equations, basic concept of numerical methods, justification of the study, aims and objectives of the study, the methodology of the study, expected contributions to knowledge and limitations to the study. In Chapter two, relevant and related literature are reviewed. Chapter three contains detailed discussions on the methodology and derivation of the methods. Chapter four contains analysis of basic properties of the methods developed as well as an investigation of their weak stability properties. In Chapter five, sample problems are used to test the performance of the one step methods developed and the computed solutions are compared with the exact solutions of the sample problems and the results from existing linear multistep methods. The results are also discussed in this chapter. Finally, Chapter six contains summary, conclusion, recommendations and contributions to the body of knowledge. Open problems have also been suggested, followed by references and appendices.

In what follows, the existence and uniqueness of the solutions of higher order ordinary differential equations is discussed.

1.2 Existence and Uniqueness of Solutions of Initial Value Problems of Ordinary Differential Equations

In this section, existence and uniqueness theorem by Wend (1967) is adopted to establish the existence and uniqueness of solutions of (1.1.1). The proof of the theorem can be found in Wend (1967 and 1969).

Theorem 1.2.1

Let R be a region defined by the inequalities $0 \leq x - x_0 < a$, $|s_k - y_k| < b_k$, $k = 0, 1, \ldots, n-1$, where $y_k \geq 0$ for $k > 0$. Suppose the function $f(x, s_0, s_1, \ldots, s_{n-1})$ in (1.1.1) is nonnegative, continuous and nondecreasing in x, and continuous and nondecreasing in s_k for each $k = 0, 1, \ldots, n-1$ in the region R. If in addition $f(x, y_0, \ldots, y_{n-1}) \neq 0$ in R for $x > x_0$ then, the initial value problem (1.1.1) has at most one solution in R. (Wend, 1967)

1.3 Basic Features of Numerical Methods

In this section, basic concepts encountered in this work are defined.

Definition 1.3.1

Consider the sequence of points $\{x_n\}$ in the interval $I = [a, b]$ defined by $a = x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$ such that $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, \ldots, N-1$. The parameter h_i is called the step size (or Mesh size).

1.3.1 One Step Methods

One step methods are methods that use data at a single point, say point n, to advance the solution to point $n+1$. Conventionally, one step numerical integrators for initial value problems are described as

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \tag{1.3.1}$$

where $\phi(x_n, y_n; h)$ is the increment function and h is the step size adopted in the subinterval $[x_n, x_{n+1}]$.

The methods can be formulated in explicit form, in which case the increment function is defined as in (1.5.1) or in implicit form where the increment function is defined in terms of the independent variable as $\phi(x_n, y_n, y_{n+1}; h)$.

Some examples of one step methods include the backward and forward Euler's methods; the midpoint method; the modified midpoint method; the trapezoid and modified trapezoid methods (otherwise called the modified Euler's method;) and the Runge-Kutta methods. Of all the single step methods, the fourth-order Runge-Kutta method is the most popular.

Implicit one step methods, (Morisson and Stoller (1958), Ceschino and Kuntzmann (1963) and Butcher (1964)), are of much interest in the development of the methods proposed in this work .

1.3.2 Linear Multistep Methods (LMMs)

Unlike the one step methods considered in the previous section where only a single value y_n was required to compute the next approximation y_{n+1}, LMM need two or more preceding values to be able to calculate y_{n+1}.

Given a sequence of equally spaced grid points $x_{n+j}, j = 0, 1, 2, \ldots, k-1$ with step size h, let y_{n+j} be an approximation to the theoretical solution of (1.1.1) at x_{n+j}, that is, $y(x_{n+j})$ and $f_{n+j} \cong f(x_{n+j}, y_{n+j}, y'_{n+j}, \ldots, y^{(\mu-1)}_{n+j})$.

Definition 1.3.2

A general k-step linear multistep method is defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^{\mu} \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.3.2}$$

where the coefficients $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_k$ are real constants and $y_{n+j} = y(x_{n+j})$ and $f_{n+j} = f(x_{n+j}., y_{n+j}, y'_{n+j}, \ldots, y^{(\mu-1)}_{n+j})$. (Lambert 1991)

Remark

(1) In order to avoid degenerate cases, we shall assume that $\alpha_k \neq 0$ and that α_0 and β_0 are not both equal to zero.

(2) If $\beta_k = 0$ then y_{n+k} is obtained explicitly from previous values of y_{n+j} and f_{n+j}, and the k-step method is said to be explicit.

On the other hand, if $\beta_k \neq 0$ then y_{n+k} appears not only on the left-hand side of (1.3.2) but also on the right within f_{n+k}; due to this implicit dependence on y_{n+k}, the method is then called implicit.

(3) The k-step LMM (1.3.2) is called linear because it involves only linear combinations of the y_{n+j} and the f_{n+j}.

(4) For the purpose of this work, the coefficients α_j's and β_j's in (1.3.2) are considered as real and continuous. In this case, (1.3.2) is referred to as Continuous Linear Multistep Methods,(CLMMs), (Awoyemi, 1992).

1.3.3 Hybrid Methods

Continuous Linear Multistep Methods (CLMMs) when compared to Runge-Kutta methods have the advantage of being more efficient in terms of accuracy and weak stability properties for a given number of function evaluations per step, but have the disadvantage of requiring starting values and special procedures for changing step sizes. The difficulties could be addressed if the step number of the CLMMs is reduced, the only obstacle to this is in satisfying the "zero stability barrier" of Dahlquist (1959 and 1963). This barrier implies that a zero stable CLMMs is at best of order $p = k+1$ for k odd and of order $p = k + 2$ for even k. Incorporating function evaluation at offstep points affords the opportunity of circumventing the "zero stability barrier". According to Lambert (1973), this technique was used independently by Gragg and Stetter (1964), Gear (1964) and later by Butcher (1965). The beauty of this method, which was named "Hybrid methods" by Gear (1964), is that while retaining certain characteristics of CLMMs, hybrid methods share with Runge-Kutta methods the property of utilizing data at off step points and the flexibility of changing step length.

Definition 1.3.3

A k-step hybrid formula is defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h' \beta_\nu f_{n+\nu} \tag{1.3.3}$$

where $\alpha_k = +1$, α_0 and β_0 are both not zero, $\nu \notin \{0, 1, \ldots, k\}$, $y_{n+j} = y(x_n + jh)$ and $f_{n+\nu} = f(x_{n+\nu}, y_{n+\nu})$. (Lambert, 1973)

Remark

For the purpose of this work, the coefficients $\alpha_j, \beta_j,\ \beta_\nu$ and α_ν will be real and continuous functions and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$.

1.3.4 Block Methods

A block method is formulated in terms of linear multistep methods. It preserves the traditional advantage of one step methods, of being self-starting and permitting easy change of step length (Lambert, 1973). Their advantage over Runge-Kutta methods lies in the fact that they are less expensive in terms of the number of functions evaluation for a given order. The method generates simultaneous solutions at all grid points.

According to Chu and Hamilton (1987) a block method can be defined as follows:

Definition 1.3.4

Let Y_m and F_m be defined by $Y_m = (y_n, y_{n+1}, \ldots, y_{n+r-1})^T$, $F_m = (f_n, f_{n+1}, \ldots, f_{n+r-1})^T$. Then a general k-block, r-point block method is a matrix of finite difference equation of the form

$$Y_m = \sum_{j=1}^{k} A_i Y_{m-i} + h \sum_{i=0}^{k} B_i F_{m-i} \tag{1.3.4}$$

where all the A_i's and B_i's are properly chosen $r \times r$ matrix coefficients and $m = 0, 1, 2, \ldots$ represents the block number, $n = mr$ is the first step number of the mth block and r is the proposed block size.

Remark

In the sequel, (1.3.4) will be redefined to suit the purpose of this research later in Chapter three.

1.4 Statement of the Problem

Conventional methods of solving higher order IVPs of general ODEs by reduction order method has been reported to have setbacks such as computational burden, complication in writing computer programs and resultant wastage of computer time (Awoyemi, 1992) and the inability of the method to utilize additional information associated with a specific ODEs such as the oscillatory nature of the solution (Vigo-Aguiar and Ramos, 2006) occasioned by the increased dimension of the problem and the low order of accuracy of the methods employed to solve the system of first-order IVPs of ODEs.

Equivalently, linear multistep methods implemented by the predictor-corrector mode have been found to be very expensive to implement in terms of the number of function evaluations per step, the predictors often have lower order of accuracy than the correctors especially when all the step and offstep points are used for collocation and interpolation.

The application of hybrid methods in the linear multistep methods, to achieve reduction in the step number, in the predictor corrector mode is compounded by the need to develop predictors for the evaluation of the corrector at offstep points making the approach even more tedious and time consuming (Lambert, 1991).

The introduction of block methods to cushion the challenges associated with linear multistep methods implemented in the predictor-corrector mode has largely been concentrated in solving IVPs of special ODEs.

In view of the foregoing, this research is motivated by the need to address the set backs associated with the existing methods, by developing a method that harnesses the beautiful properties of these existing methods. Such a method would be less expensive, in terms of the number of functions evaluation per step; highly efficient in terms of accuracy and error term; flexible in change of step size; possess better rate of convergence and weak stability properties; and very easy to program resulting in economy of computer time.

1.5 Aims and Objectives

The aim of this research is to develop continuous implicit hybrid one step methods for the direct solution of initial value problems of general second order ordinary differential equations. To achieve this aim, the following objectives were outlined:

(i) to develop continuous implicit one step methods by collocating and interpolating at both the step and offstep points;

(ii) to implement the methods without the rigor of developing predictors separately;

(iii) to analyse basic properties of the method developed which include order, consistency, zero stability, convergence and region of absolute stability;

(iv) to write computer programs, that are easy to implement; and

(v) to test the performance of the new methods for accuracy and efficiency.

1.6 Research Methodology

Power series polynomial of the form

$$y(x) = \sum_{j=0}^{k} a_j x^j \tag{1.7.1}$$

is used as a basis function to approximate the solution of the initial value problems of general second order ordinary differential equation of the form

$$y'' = f(x, y, y'), \; y(a) = y_0, \;\; y'(a) = y_1 \tag{1.7.2}$$

on the interval $[a, b]$, (Awoyemi, 1995). Equation (1.8.2) was collocated at all grid points and interpolated at selected grid points after offstep points are introduced to allow the application of continuous linear multistep procedure. The resulting system of equations are then solved by Gaussian elimination method to obtain the unknown parameters. By substituting these parameters back into (1.8.1), a continuous implicit hybrid one step method is obtained in the form of a continuous linear multistep method. A modified Block method is then employed to implement the new method. Computer programs were written using FORTRAN 95/2003 programming language to test the performance of the methods. The basic properties such as consistency and zero stability are analyzed to determine the convergence of the methods.

1.7 Contribution to Knowledge

The following contributions are made to the body of knowledge:

(i) a new class of continuous implicit hybrid one step method for the direct solutions of initial value problems of general second order ordinary differential equations has been developed;

(ii) a new formula for block hybrid methods for the direct solution of initial value problems of second order ordinary differential equations is introduced; and

(iii) very accurate and highly efficient computer codes have been written for the implementation of the new methods.

1.8 Limitations of the Study

The research is limited to the following:

(i) only continuously differentiable functions in the interval of integration were considered;

(ii) the basis function considered in this work is the power series polynomial in view of its smoothness;

(iii) the research work adopted only continuous hybrid linear multistep methods where the step number $k = 1$;

(iv) only implicit block methods were adopted in this research work.

In the next chapter, some literatures on existing numerical methods for solving IVPs of higher order ordinary ordinary differential equations were reviewed.

Chapter 2

Literature Review

2.1 Introduction

The desire to obtain more accurate approximate solutions to mathematical models, arising from science, engineering and even social sciences, in the form of ordinary differential equations which do not have analytical solutions, has led many scholars to propose several different numerical methods.

In this chapter, some of the many contributions available in the literature are reviewed. Specifically, those numerical methods for the solution of (1.1.1) and in particular the special case (1.8.2), when $n = 2$ is considered.

2.2 Review of Existing Methods

In most applications, (1.1.1) is solved by reduction to an equivalent system of first order ordinary differential equations of the form

$$y' = f(x, y), \quad y(a) = \mu; a \leq x \leq b; x, y \in \mathbb{R}^n \text{ and } f \in C^1[a, b] \tag{2.2.1}$$

for any appropriate numerical method to be employed to solve the resultant system. The approach is extensively discussed by some prominent authors such as Lambert (1973, 1991), Goult, Hoskins,and Pratt (1973), Lambert and Watson (1976), Dodes (1978), Jain, Kambo and Rakesh (1984), Ixaru (1984), Kadalbajoo and Raman (1986), Jacques and Judd (1987), Fatunla (1988), Sarafyan (1990), Bun and Vasil'Yer

(1992), Awoyemi (1992), Onumanyi, Awoyemi, Jator, and Sirisena (1994), Brugnano and Trigiante (1998), Jator (2001), Juan (2001), among others. In spite of the success of this approach, there are setbacks. For example, writing computer programs for these methods is often cumbersome especially when subroutines are incorporated to supply starting values required for the methods. The consequences are in longer computer time and more human effort, (Awoyemi,1992). In addition, this method does not utilize additional information associated with specific ordinary differential equations, such as the oscillatory nature of the solution, (Vigo-Aguilar and Ramos, 2006). Furthermore, according to Bun and Vasil'Yer (1992), a more serious disadvantage of the method is the fact that the given system of equations to be solved cannot be solved explicitly with respect to the derivatives of the highest order, (Kayode, 2004). For these reasons, this method is inefficient and not suitbale for general purpose applications.

Rutishauser (1960), examined the direct solution of (1.1.1) and its equivalent first order initial value problems and concluded that the choice of approach depends on the particular problem under consideration. Many other Scholars such as Henrici (1962), Gear (1971), Hairer and Wanner (1976), Jeltsch (1976), Twizel and Khaliq (1984), Chawla and Sharma (1985), Fatunla (1988), Taiwo and Onumanyi (1991), Awoyemi (1995, 1998, 1999,2001, 2003, 2005), Simos (2002), Onumanyi Sirisena and Chollom (2001), Awoyemi and Kayode (2005), Kayode (2004, 2005 and 2009), and Yusuph and Onumanyi (2005), Vigo-Aguiar and Ramos (2006), Adesanya, Anake and Udoh (2008), Adesanya, Anake and Oghoyon (2009), etc, suggested in the literature that a better alternative is to solve (1.1.1) directly without first reducing it to a system of first order ordinary differential equations.

In particular, this work is concerned only with the direct solutions of (1.1.1) for $n = 2$ without reducing it to an equivalent system of first order equations. However, many Scholars such as Henrici (1962), Jeltsch (1976), Twizel and Khaliq (1984), Awoyemi (1998), Simos (2002), and Yusuph and Onumanyi (2005), have devoted a lot of attention to the development of various methods for solving directly the special second order initial value problems of the form

$$y'' = f(x, y), \quad y(a) = \mu_0, \quad y'(a) = \mu_1, \tag{2.2.2}$$

which is the mathematical formulation for systems without dissipation. Nystrom for instance, considered a step-by-step method based on the classical Runge-Kutta methods, (Fatunla, 1988). Later, Hairer and Wanner (1976) developed Nystrom-type methods for (2.2.2) in which they listed order conditions for the determination of the parameters of the method. Gear (1971), Hairer (1979), Chawla and Sharma (1981), independently developed explicit and implicit Runge-Kutta Nystrom type methods. Dormand and Prince (1987) also developed two classes of embedded Runge-Kutta-Nystrom methods for the direct solution of (2.2.2). First step methods were also discussed by Gonzalez and Thompson (1997) as starting values required to implement the Numerov method for the direct solution of (2.2.2).

In the literature also, Henrici (1962) and later Lambert (1973) postulated the derivation of linear multistep methods with constant coefficients for solving (2.2.2). Fatunla (1984, 1985, 1988) developed P-stable one-leg constant coefficients linear multistep method in which Pade approximation was used to realize his methods for the solution of (2.2.2). Vigo-Aguilar and Ramos (2006) in their contribution discussed variable step size multistep schemes based on the Falkner method and directly applied it to eqn.(2.2.2) in predictor-corrector mode. More on LMM can be found in Lie and Norsett (1989), Enright (1974), Dahlquist (1978), Enright and Addison (1984), Chawla and Rao (1985), Chawla and McKee (1986), to mention a few. The procedure they adopted for this class of methods is such that the resultant methods are not continuous, and therefore it is impossible to find the first and higher order derivatives of y with respect to x; and so the scope of this class of methods is limited in application, (Awoyemi, 2001).

Onumanyi, Awoyemi, Jator and Sirisena (1994), Awoyemi (1992, 1995, 1999, 2001). Onumanyi, Sirisena and Jator (1999), Awoyemi and Kayode (2005), Kayode (2004 and 2005), proposed linear multistep methods with continuous coefficient for initial value problems of the form (1.8.2) in the predictor corrector mode based on collocation method using power series polynomial as the basis function and Taylor series algorithm to supply starting values. According to Awoyemi (1992), continuous linear multistep methods have greater advantages over the discrete methods in that they give better error estimates, provide a simplified form of coefficients for further analytical work at different points and guarantee easy approximation of solutions at all interior points of the integration interval.

In spite of these advantages, the continuous linear multistep methods, like the constant coefficients linear multistep methods, are usually applied to the initial value problems as a single formula and this has some inherent disadvantages. For instance, they require the use of known pivotal points generated through the use of a set of so-called pivot formulas which are known as predictors, (Sarafyan, 1965). Implementation of the method in predictor-corrector mode is very costly as subroutines are very complicated to write because of the special techniques required to supply starting values and for varying the step size which leads to longer computer time and more human effort.

Another method that has been proposed in the literature is the hybrid method. This method while retaining certain characteristics of the continuous linear multistep methods, share with the Runge-Kutta methods the property of utilizing data at other points other than the step points $\{x_{n+j}; x_{n+j} = x_n + jh\}$. The method is useful in reducing the step number of a method and still remains zero stable. According to Lambert (1973), hybrid method was first introduced independently by Gragg and Stetter (1964), Gear (1964) and Butcher (1965). Hairer (1979) later used Pade approximation to develop a fourth-order P-stable hybrid method for solving eqn.(2.2.2) using one offpoint. In the same spirit as Hairer, Chawla (1981) and Cash (1981) independently showed that the zero stability barrier imposed by Lambert (1973) and Dahlquist (1978) could indeed be circumvented by considering two step hybrid methods. The result of their exposition was the development of fourth and sixth order P-stable methods. Fatunla (1984) also used Pade approximation to develop one-leg hybrid multistep method. However, Jain *et al.* (1984) in developing their sixth-order symmetric multistep method for period IVP of type (2.2.2), observed that the cost of implementing the method by Cash (1981) was high due to many function evaluations per iteration. Awoyemi (1995) adopted the method and proposed a two-step hybrid multistep method with continuous coefficients for the solution of (2.2.2) based on collocation at selected grid points and using off-grid points to upgrade the order of the method and to provide one additional interpolation point and implemented on the hybrid predictor-corrector mode. Later, Adee, Onumanyi, Sirisena and Yahaya (2005) in solving (2.2.2) used hybrid formula of order four to generate starting values for Numerov method. D'Ambrosio, Ferro and Paternoster (2009) on the other hand proposed a two step hybrid collocation method based on Butcher's general linear methods (GLM) to solve (2.2.2). Other Scholars who have studied hybrid methods include; Onumanyi *et al.* (2001), Yahaya and Badmus (2009), etc. According to Lambert (1973), hybrid method is not a method in its own right since special predictors were required to estimate the solution at the offstep point and the derivative function as well.

In view of all the disadvantages mentioned above, many researchers concentrated efforts on advancing the numerical solution of initial value problems of ordinary differential equations. One of the outcomes is the development of a class of methods called Block method. The method simultaneously generates approximations at different grid points in the interval of integration and is less expensive in terms of the number of function evaluations compared to the linear multistep methods or Runge-

Kutta methods. This method was first proposed by Milne (1953), who advocated their use only as a means of obtaining starting values for predictor-corrector algorithm. This was considered in the same light by Sarafyan (1965) however, Rosser (1967) later developed Milne's proposal to algorithms suitable for general use. Using Rosser's approach, Lambert (1973), developed a two-step fourth-order explicit block method. Earlier on, implicit block methods had been proposed. For instance, an example due to Clippinger and Dimsdale in Grabbe, Ramo and Woolridge (1958) was analyzed by Shampine and Watts (1969) as implicit one step block method. Since then, many contributions on block methods with different approaches have been proposed in the literatures in recent years. For instance, Chu and Hamilton (1987) suggested a generalization of the linear multistep method to a class of multi-block methods where step values are all obtained together in a single block advance accomplished by allocating the parallel tasks on separate processors. Fatunla (1991 and 1994) proposed block method for the solutions of special second order ordinary differential equations which was later developed by Omar and Suleiman (1999, 2003 and 2005) to obtain explicit and implicit parallel block methods for solving higher order ordinary differential equations where the derivative function is approximated by a suitable interpolating polynomial within a specified interval of integration. This method was adopted by Ismail, Ken and Othman (2009) to develop explicit-implicit three-point block method for the direct solution of special second order ordinary differential equations. Many other scholars such as Majid, Suleiman and Omar (2006), Majid and Suleiman (2007), Majid, Azimi and Suleiman (2009), Ibrahim, Suleiman and Othman (2009), etc. have adopted block methods where the derivative function was interpolated using Lagrange interpolation. In another approach adopted to implement implicit block methods however, the need to generate predictors is still required. For example Yahaya (2007), Awoyemi, Adesanya and Ogunyebi (2009) and Adesanya, *et al.* (2008) and Adesanya *at al.* (2009) used the forward difference method, the Newton's forward difference method and Newton's polynomials, respectively, to generate predictors for Fatunla's block method in order to solve (1.8.2). These methods have largely focused on solving only special type ordinary differential equations with very few attempts in favour of (1.8.2).

Recently, Jator (2007) and Jator and Li (2009) have proposed five-step and four-step self-starting methods which adopt continuous linear multistep method to obtain finite difference methods applied respectively as a block for the direct solution of (1.8.2).

These different methods have their very desirable qualities. Thus, the method proposed in this research is one that combines these desirable qualities for the direct solution of (1.8.2).

In the next chapter, the methodology of our work is presented and the derived methods are specified.

Chapter 3

Methodology

3.1 Introduction

This chapter describes the development of continuous implicit hybrid one step methods for the solutions of IVPs of higher order ODE. The idea is to approximate the exact solution $y(x)$ of (1.8.2) in the partition $\pi_{[a,b]} = [a = x_0 < x_1 < \cdots < x_n = b]$ of the integration interval $[a, b]$ by a power series polynomial of the form;

$$p(x) = \sum_{j=0}^{\infty} a_j x^j \tag{3.1.1}$$

where $a_j \in \mathbb{R}$, $y \in C^{\infty}(a, b)$.

The method is derived by the introduction of offstep points in the conventional one step scheme following the method of Gragg and Stetter (1964), Gear (1964), Butcher (1965), Kohfield (1967), Brush, Kohfield and Thompson (1967) and recently Awoyemi and Idowu (2005). Then, (3.1.1) is interpolated at selected grid points chosen according to the Stormer-Cowell method. The second derivative of (3.1.1) is substituted into (1.8.2) to obtain a differential system which is evaluated, respectively, at the step and offstep points. Using this technique, in the form of linear multistep methods, accurate continuous implicit hybrid one step methods are obtained. Finally, methods obtained are implemented by the application of a modification of the implicit one step block method proposed by Shampine and Watts (1969). This modification caters for the offstep points and y'.

In the sections that follow, the derivation of five different continuous implicit one step methods with varying number of 'offstep' points are outlined.

3.2 Derivation of Methods

Let the approximate solution be given as a power series of a single variable x in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{3.2.1}$$

with the second derivative given by

$$y''(x) = \sum_{j=0}^{r+s-1} j(j-1)a_j x^{j-2} \tag{3.2.2}$$

where $x \in [a, b]$, the a's are real unknown parameters to be determined and $r + s$ is the sum of the number of collocation and interpolation points. Let the solution of (1.8.2) be sought on the partition

$$\pi_N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$$

of the integration interval $[a, b]$ with a constant step size h, given by

$$h = x_{n+1} - x_n, \quad n = 0, 1, \ldots, N.$$

Then, substituting (3.2.2) in (1.8.2) gives

$$\sum_{j=0}^{r+s-1} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3.2.3}$$

Now interpolating (3.2.1) at $x_{n+s}, s = 0, \nu_i$ and collocating (3.2.3) at $x_{n+r}, r = 0, \nu_i, k$, where r, s and i represent the number of collocation, interpolation and offstep points respectively and k is the step number, leads to the following system of equations

$$\sum_{j=0}^{r+s-1} j(j-1)a_j x^{j-2} = f_{n+r}, r = 0, \nu_i, k, \quad i = 1, 2, \ldots, m \tag{3.2.4}$$

$$\sum_{j=0}^{r+s-1} a_j x_{n+s}^j = y_{n+s}, \quad s = 0, \nu_i, \quad i = 1, \ldots, m \tag{3.2.5}$$

(3.2.4) and (3.2.5) can be combined to form a matrix as follows

$$\begin{bmatrix} x_n^0 & x_n^1 & x_n^2 & x_n^3 & \cdots & x^N \\ x_{n+\nu_1}^0 & x_{n+\nu_1}^1 & x_{n+\nu_1}^2 & x_{n+\nu_1}^3 & \cdots & x_{n+\nu_1}^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n+\nu_m}^0 & x_{n+\nu_m}^1 & x_{n+\nu_m}^2 & x_{n+\nu_m}^3 & \cdots & x_{n+\nu_m}^N \\ 0 & 0 & 2x_n^0 & 6x_n^1 & \cdots & N(N-1)x_N^{N-2} \\ 0 & 0 & 2x_{n+\nu_1}^0 & 6x_{n+\nu_1}^1 & \cdots & N(N-1)x_{n+\nu_1}^{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2x_{n+\nu_m}^0 & 6x_{n+\nu_m}^1 & \cdots & N(N-1)x_{n+\nu_m}^{N-2} \\ 0 & 0 & 2x_{n+1}^0 & 6x_{n+1}^1 & \cdots & N(N-1)x_{n+1}^{N-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{r+s-1} \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\nu_1} \\ \vdots \\ y_{n+\nu_m} \\ f_n \\ f_{n+\nu_1} \\ \vdots \\ f_{n+\nu_m} \\ f_{n+1} \end{bmatrix} \tag{3.2.6}$$

where $\nu_i \in (x_0, x_{n+1}), i = 1, \cdots, m$.

Using Gaussian elimination method, (3.2.6) is solved for the a_j's. The values of the a_j's obtained are then substituted into (3.2.1) to give, after some manipulations, a continuous hybrid one step method in the form of the continuous linear multistep method

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{\nu_i} \alpha_{\nu_i}(x) y_{n+\nu_i} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \sum_{\nu_i} \beta_{\nu_i}(x) f_{n+\nu_i} \right], \tag{3.2.7}$$

where $y_{n+j} = y(x_{n+j})$ and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$ and μ, is the order of the problem.

In what follows, let us express $\alpha_j(x)$ and $\beta_j(x)$ as continuous functions of t by letting

$$t = \frac{x - x_{n+\nu_i}}{h} \tag{3.2.8}$$

and noting that

$$\frac{dt}{dx} = \frac{1}{h}.$$

The derivative of (3.2.7) is given by

$$y'(x) = \frac{1}{h}\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+\frac{1}{h}\sum_{\nu_i}\alpha_{\nu_i}(x)y_{n+\nu_i}+h\left[\sum_{j=0}^{k}\beta_j(x)f_{n+j}+\sum_{\nu_i}\beta_{\nu_i}(x)f_{n+\nu_i}\right] \quad (3.2.9)$$

To implement (3.2.7), we use a modified block method defined as follows;

$$h^{\lambda}\sum_{j=1}^{q}a_{ij}y_{n+j}^{\lambda} = h^{\lambda}\sum_{j=0}^{q}e_{ij}y_n^{\lambda} + h^{\mu-\lambda}\left[\sum_{j=1}^{q}d_{ij}f_n + \sum_{j=1}^{q}b_{ij}f_{n+j}\right] \quad (3.2.10)$$

where λ is the power of the derivative of the continuous method and μ is the order of the problem to be solved; $q = r + s$.

In vector notation, (3.2.10) can be written as

$$h^{\lambda}\bar{a}Y_m = h^{\lambda}\bar{e}y_m + h^{\mu-\lambda}\left[\bar{d}f(y_m) + \bar{b}F(Y_m)\right] \quad (3.2.11)$$

The matrices $\bar{a} = (a_{ij})$, $\bar{b} = (b_{ij})$, $\bar{e} = (e_{ij})$, $\bar{d} = (d_{ij})$ are constant coefficient matrices and $Y_m = (y_{n+\nu_i}, y_{n+1}, y'_{n+\nu_i}, y'_{n+1})^T$, $y_m = (y_{n-(r-1)}, y_{n-(r-2)}, \ldots, y_n)^T$, $\bar{F}(Y_m) = (f_{n+\nu_i}, f_{n+j})^T$ and $f(y_m) = (f_{n-i}, \ldots, f_n)$, $i = 1, \ldots, q$. The normalized version of (3.2.11) is given by

$$\bar{A}Y_m = h^{\lambda}\bar{E}y_m + h^{\mu-\lambda}[\bar{D}f(y_m) + \bar{B}F(Y_m)] \quad (3.2.12)$$

The methods obtained are specified in the next section.

Note that since we are developing a one step method, throughout this thesis the step number k will always be one (i.e. $k = 1$).

3.3 Specification of the Methods

3.3.1 One Step Method with One Offstep Point

To derive this method, one 'offstep' point is introduced. This offstep point is carefully selected to guarantee zero stability condition. For this method, the offstep point is

$\nu_1 = \frac{1}{2}$. Using (3.2.1) with $r = 3$ and $s = 2$, we have a polynomial of degree $r + s - 1$ as follows:

$$y(x) = \sum_{j=0}^{4} a_j x^j \tag{3.3.1}$$

with second derivative given by

$$y''(x) = \sum_{j=0}^{4} j(j-1)a_j x^{j-2} \tag{3.3.2}$$

Substituting (3.3.2) into (1.8.2) gives

$$\sum_{j=0}^{4} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3.3.3}$$

Now collocating (3.3.3) at x_{n+r}, $r = 0, \frac{1}{2}$ and 1, and interpolating (3.3.1) at x_{n+s}, $s = 0, \frac{1}{2}$ leads to a system of equations written in the matrix form $AX = B$ as

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \tag{3.3.4}$$

Equation (3.3.4) is solved by Gaussian elimination method to obtain the value of the unknown parameters a_j, $(j = 0, 1, \cdots, 4)$ as follows:

$$\begin{aligned} a_0 &= y_n - a_1 x_n - a_2 x_n^2 - a_3 x_n^3 - a_4 x^4 \\ a_1 &= \frac{y_{n+\frac{1}{2}} - y_n}{\frac{1}{2}h} - a_2(x_{n+\frac{1}{2}} - x_n) - a_3(x_{n+\frac{1}{2}}^2 + x_{n+\frac{1}{2}}x_n + x_n^2) \\ a_2 &= \frac{1}{2}f_n - 3a_3 x_n - 6a_4 x_n^2 \\ a_3 &= \frac{f_{n+\frac{1}{2}}}{3h} - 2a_4(x_{n+\frac{1}{2}} + x_n) \\ a_4 &= \frac{\frac{1}{2}f_{n_1} - f_{n+\frac{1}{2}} + \frac{1}{2}f_n}{3h^2} \end{aligned} \tag{3.3.5}$$

Substituting (3.3.5) into (3.3.1) yields a continuous implicit hybrid one step method in the form of a continuous linear multistep method described by the formula

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{\nu_i} \alpha_{\nu_i}(x) y_{n+\nu_i} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \sum_{\nu_i} \beta_{\nu_i}(x) y_{n+\nu_i} \right] \quad (3.3.6)$$

where, for $k = 1$, $i = 1$ and $\nu_1 = \frac{1}{2}$, yields the parameters α_j and β_j, $j = 0, \nu_1, 1$ as the following continuous functions of t

$$\begin{aligned} \alpha_0(t) &= -2t \\ \alpha_{\frac{1}{2}}(t) &= 2t + 1 \\ \beta_0(t) &= \frac{h^2}{48}(8t^4 - 8t^3 + 3t) \\ \beta_{\frac{1}{2}}(t) &= \frac{h^2}{24}(-8t^4 + 12t^2 + 5t) \\ \beta_1(t) &= \frac{h^2}{48}(8t^4 + 8t^3 - t) \end{aligned} \quad (3.3.7)$$

Using (3.3.7) for $x = x_{n+1}$ and $i = 1$ so that $t = \frac{1}{2}$, (3.3.6) reduces to

$$y_{n+1} - 2y_{n+\frac{1}{2}} + y_n = \frac{h^2}{48}\left[f_{n+1} + 10f_{n+\frac{1}{2}} + f_n\right] \quad (3.3.8)$$

Differentiating (3.3.7) yields

$$\begin{aligned} \alpha_0'(t) &= \frac{-2}{h} \\ \alpha_{\frac{1}{2}}'(t) &= \frac{2}{h} \\ \beta_0'(t) &= \frac{h}{48}(32t^3 - 24t^2 + 3) \\ \beta_{\frac{1}{2}}'(t) &= \frac{h}{48}(-32t^3 + 24t + 5) \\ \beta_1'(t) &= \frac{h}{48}(32t^3 + 24t^2 - 1) \end{aligned} \quad (3.3.9)$$

On evaluating (3.3.9) at $x = x_n$,$x_{n+\frac{1}{2}}$ and x_{n+1} respectively, using (3.2.8) so that $t = -\frac{1}{2}, 0, \frac{1}{2}$, the following discrete methods are obtained

$$\begin{aligned} 48hy'_n - 96y_{n+\frac{1}{2}} + 96y_n &= h^2\left[f_{n+1} - 6f_{n+\frac{1}{2}} - 7f_n\right] \\ 48hy'_{n+\frac{1}{2}} - 96y_{n+\frac{1}{2}} + 96y_n &= h^2\left[-f_{n+1} + 10f_{n+\frac{1}{2}} + 3f_n\right] \\ 48hy'_{n+1} - 96y_{n+\frac{1}{2}} + 96y_n &= h^2\left[9f_{n+1} + 26f_{n+\frac{1}{2}} + f_n\right] \end{aligned} \tag{3.3.10}$$

The modified block formulae (3.2.11) and (3.2.12) are employed to simultaneously obtain values for $y_{n+\frac{1}{2}}$, y_{n+1}, $y'_{n+\frac{1}{2}}$ and y'_{n+1} needed to implement (3.3.8). Now, combining (3.3.8) and (3.3.10) in the form of (3.2.11) and (3.2.12) yield the block method

$$\begin{bmatrix} -96 & 48 & 0 & 0 \\ -96 & 0 & 0 & 0 \\ -96 & 0 & 48h & 0 \\ -96 & 0 & 0 & 48h \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} -48 & 0 \\ -96 & 48h \\ -96 & 0 \\ -96 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \begin{bmatrix} h^2 \\ -7h^2 \\ 3h^2 \\ h^2 \end{bmatrix} [f_n]$$

$$+ \begin{bmatrix} 10h^2 & h^2 \\ -6h^2 & h^2 \\ 10h^2 & -h^2 \\ 26h^2 & 9h^2 \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \tag{3.3.11}$$

Using (3.2.12), we obtain the block solution

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 10 & \frac{1}{2}h \\ 1 & h \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \begin{bmatrix} \frac{7}{96}h^2 \\ \frac{1}{6}h^2 \\ \frac{5}{24}h \\ \frac{1}{6}h \end{bmatrix} [f_n]$$

$$+\begin{bmatrix} \frac{1}{16}h^2 & -\frac{1}{96}h^2 \\ \frac{1}{3}h^2 & 0 \\ \frac{1}{3}h & -\frac{1}{24}h \\ \frac{2}{3}h & \frac{1}{6}h \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix} \tag{3.3.12}$$

Equation (3.3.12) can be written explicitly as

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + \frac{7}{96}h^2 f_n + \frac{h^2}{96}\left[-f_{n+1} + 6f_{n+\frac{1}{2}}\right] \\ y_{n+1} &= y_n + hy'_n + \frac{1}{6}h^2 f_n + \frac{1}{3}h^2 f_{n+\frac{1}{2}} \\ y'_{n+\frac{1}{2}} &= y'_n + \frac{5}{24}hf_n + \frac{h}{24}\left[-f_{n+1} + 8f_{n+\frac{1}{2}}\right] \\ y'_{n+1} &= y'_n + \frac{1}{6}hf_n + \frac{h}{6}\left[f_{n+1} + 4f_{n+\frac{1}{2}}\right] \end{aligned} \tag{3.3.13}$$

3.3.2 One step method with Two Offstep Points

In this case, two offstep points are introduced. Similarly these points are carefully selected to guarantee the zero stability of the method. Here, $i = 2$ so that, $\nu_1 = \frac{1}{3}$, $\nu_2 = \frac{2}{3}$, $r = 4$ and $s = 2$.

From (3.2.1) for $r = 4$ and $s = 2$, we obtain the polynomial of degree $r + s - 1$ as follows;

$$y(x) = \sum_{j=0}^{5} a_j x^j \tag{3.3.14}$$

with second derivative given by

$$y'' = \sum_{j=0}^{5} j(j-1)a_j x^{j-2} \tag{3.3.15}$$

Substituting (3.3.15) into (1.8.2) gives

$$\sum_{j=0}^{5} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3.3.16}$$

Collocating (3.3.16) at x_{n+r}, $r = 0, \frac{1}{3}, \frac{2}{3}$ and 1 and interpolating (3.3.14) at x_{n+s}, $s = \frac{1}{3}$ and $\frac{2}{3}$ leads to a system of equations written in matrix form $AX = B$ as follows

$$\begin{bmatrix} 1 & x_{n+\frac{1}{3}} & x^2_{n+\frac{1}{3}} & x^3_{n+\frac{1}{3}} & x^4_{n+\frac{1}{3}} & x^5_{n+\frac{1}{3}} \\ 1 & x_{n+\frac{2}{3}} & x^2_{n+\frac{2}{3}} & x^3_{n+\frac{2}{3}} & x^4_{n+\frac{2}{3}} & x^5_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x^2_{n+\frac{1}{3}} & 20x^3_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x^2_{n+\frac{2}{3}} & 20x^3_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{bmatrix} \qquad (3.3.17)$$

Solving (3.3.17) by Gaussian elimination method yields the a_j's as follows

$$\begin{aligned} a_0 &= y_{n+\frac{1}{3}} - a_1 x_{n+\frac{1}{3}} - a_2 x^2_{n+\frac{1}{3}} - a_3 x^3_{n+\frac{1}{3}} - a_4 x^4_{n+\frac{1}{3}} - a_5 x^5_{n+\frac{1}{3}} \\ a_1 &= \frac{y_{n+\frac{2}{3}} - y_{n+\frac{1}{3}}}{\frac{1}{3}h} - a_2(x_{n+\frac{2}{3}} + x_{n+\frac{1}{3}}) - a_3(x^2_{n+\frac{2}{3}} + x_{n+\frac{2}{3}} x_{n+\frac{1}{3}} + x^2_{n+\frac{1}{3}}) \\ &\quad -a_4 \left(x^3_{n+\frac{2}{3}} + x^2_{n+\frac{2}{3}} x_{n+\frac{1}{3}} + x_{n+\frac{2}{3}} x^2_{n+\frac{1}{3}} + x^3_{n+\frac{1}{3}} \right) \\ &\quad -a_5 \left(x^4_{n+\frac{2}{3}} + x^3_{n+\frac{2}{3}} x_{n+\frac{1}{3}} + x^2_{n+\frac{2}{3}} x^2_{n+\frac{1}{3}} + x_{n+\frac{2}{3}} x^3_{n+\frac{2}{3}} + x^4_{n+\frac{1}{3}} \right) \qquad (3.3.18) \\ a_2 &= \frac{1}{2} f_n - 3a_3 x_n - 6a_4 x_n^2 - 10 a_5 x_n^3 \\ a_3 &= \frac{f_{n+\frac{1}{3}} - f_n}{2h} - 2a_4(x_{n+\frac{1}{3}} + x_n) - \frac{10}{3} a_5 \left(x^2_{n+\frac{1}{3}} + x_{n+\frac{1}{3}} x_n + x_n^2 \right) \\ a_4 &= \frac{\frac{1}{3} f_{n+\frac{2}{3}} - \frac{2}{3} f_{n+\frac{1}{3}} + \frac{1}{3} f_n}{\frac{8}{9} h^2} - \frac{5}{3} a_5 \left(x_{n+1} + x_{n+\frac{1}{3}} + x_n \right) \\ a_5 &= \frac{\frac{2}{27} f_{n+1} - \frac{2}{9} f_{n+\frac{2}{3}} + \frac{2}{9} f_{n+\frac{1}{3}} - \frac{2}{27} f_n}{\frac{80}{243} h^3} \end{aligned}$$

Substituting the a_js, $j = 0(1)5$ into (3.3.14) yields the continuous implicit hybrid one step method in the form of a continuous linear multistep method described by the formula

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{\nu_i} \alpha_{\nu_i} y_{n+\nu_i} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \sum_{\nu_i} \beta_{\nu_i}(x) f_{n+\nu_i} \right], \quad i = 1, 2, \qquad (3.3.19)$$

For $k = 1$, $\nu_1 = \frac{1}{3}$ and $\nu_2 = \frac{2}{3}$ and writing the α_j's and β_j's as continuous functions of t, where $t = \frac{x - x_{n+\frac{1}{3}}}{h}$, we obtain the parameters

$$
\begin{aligned}
\alpha_{\frac{1}{3}}(t) &= -3t \\
\alpha_{\frac{2}{3}}(t) &= 3t + 1 \\
\beta_0(t) &= -\frac{h^2}{1080}(243t^5 - 90t^3 + 7t) \\
\beta_{\frac{1}{3}}(t) &= \frac{h^2}{360}(243t^5 + 135t^4 - 180t^3 + 22t) \\
\beta_{\frac{2}{3}}(t) &= -\frac{h^2}{360}(243t^5 + 270t^4 - 90t^3 - 180t^2 - 43t) \\
\beta_1(t) &= \frac{h^2}{1080}(243t^5 + 405t^4 + 180t^3 - 8t)
\end{aligned}
\qquad (3.3.20)
$$

Evaluating (3.3.19) at $x = x_n$ and x_{n+1} using (3.2.8) gives values of t to be $-\frac{2}{3}$ and $\frac{1}{3}$. Thus, we obtain the discrete methods from (3.3.20) as follows

$$y_{n+1} - 2y_{n+\frac{2}{3}} + y_{n+\frac{1}{3}} = \frac{h^2}{108}\left[f_{n+1} + 10f_{n+\frac{2}{3}} + f_{n+\frac{1}{3}}\right] \qquad (3.3.21a)$$

$$y_{n+\frac{2}{3}} - 2y_{n+\frac{1}{3}} + y_n = \frac{h^2}{108}\left[f_{n+\frac{2}{3}} + 10f_{n+\frac{1}{3}} + f_n\right] \qquad (3.3.21b)$$

Differentiating (3.3.20) gives

$$
\begin{aligned}
\alpha'_{\frac{1}{3}}(t) &= -\frac{3}{h} \\
\alpha'_{\frac{2}{3}}(t) &= \frac{3}{h} \\
\beta'_0(t) &= -\frac{h}{1080}(1215t^4 - 270t^2 + 7) \\
\beta'_{\frac{1}{3}}(t) &= \frac{h}{360}(1215t^5 + 540t^3 - 540t^2 + 22) \\
\beta'_{\frac{2}{3}}(t) &= -\frac{h}{360}(1215t^4 + 1080t^3 - 270t^2 - 360t - 43) \\
\beta'_1(t) &= \frac{h}{1080}(1215t^4 + 1620t^3 + 540t^2 - 8)
\end{aligned}
\qquad (3.3.22)
$$

Evaluating (3.3.22) at $x = x_n,\ x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}}$ using (3.2.8) implies $t = -\frac{2}{3},\ -\frac{1}{3},\ 0$ and $\frac{1}{3}$ Hence, the following discrete derivative methods are obtained.

$$
\begin{aligned}
1080hy'_n - 3240y_{n+\frac{2}{3}} + 3240y_{n+\frac{1}{3}} &= h^2\left[-8f_{n+1} + 9f_{n+\frac{2}{3}} - 414f_{n+\frac{1}{3}} - 127f_n\right] \\
1080hy'_{n+\frac{1}{3}} - 3240y_{n+\frac{2}{3}} + 3240y_{n+\frac{1}{3}} &= h^2\left[7f_{n+1} - 66f_{n+\frac{2}{3}} - 129f_{n+\frac{1}{3}} + 8f_n\right] \\
1080hy'_{n+\frac{2}{3}} - 3240y_{n+\frac{2}{3}} + 3240y_{n+\frac{1}{3}} &= h^2\left[-8f_{n+1} + 129f_{n+\frac{2}{3}} + 66f_{n+\frac{1}{3}} - 7f_n\right] \\
1080hy'_{n+1} - 3240y_{n+\frac{2}{3}} + 3240y_{n+\frac{1}{3}} &= h^2\left[127f_{n+1} + 414f_{n+\frac{2}{3}} - 9f_{n+\frac{2}{3}} + 8f_n\right]
\end{aligned}
\tag{3.3.23}
$$

Combining (3.3.21) and (3.3.23) using the block formulae (3.2.11) and (3.2.12) we have, respectively,

$$
\begin{bmatrix}
-216 & 108 & 0 & 0 & 0 & 0 \\
108 & -216 & 108 & 0 & 0 & 0 \\
3240 & -3240 & 0 & 0 & 0 & 0 \\
3240 & -3240 & 0 & 1080h & 0 & 0 \\
3240 & -3240 & 0 & 0 & 1080h & 0 \\
3240 & -3240 & 0 & 0 & 0 & 1080h
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{2}{3}} \\ y'_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
108 & 0 \\ 0 & 0 \\ 0 & 1080h \\ 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_n \\ y'_n
\end{bmatrix}
$$

$$
+\begin{bmatrix}
0 \\ h^2 \\ -127h^2 \\ 8h^2 \\ -7h^2 \\ 8h^2
\end{bmatrix}
[f_n] +
\begin{bmatrix}
h^2 & 10h^2 & h^2 \\
10h^2 & h^2 & 0 \\
414h^2 & 9h^2 & -8h^2 \\
-129h^2 & -66h^2 & 7h^2 \\
66h^2 & 129h^2 & -8h^2 \\
-9h^2 & 414h^2 & 127h^2
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1}
\end{bmatrix}
\tag{3.3.24}
$$

and

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{2}{3}} \\ y'_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
1 & \frac{1}{3}h \\
1 & \frac{2}{3}h \\
1 & h \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
y_n \\ y'_n
\end{bmatrix}
+
\begin{bmatrix}
\frac{97}{3240}h^2 \\ \frac{28}{405}h^2 \\ \frac{13}{120}h^2 \\ \frac{1}{8}h \\ \frac{1}{9}h \\ \frac{1}{8}h
\end{bmatrix}
[f_n]
$$

$$
+
\begin{bmatrix}
\frac{19}{540}h^2 & -\frac{13}{1080}h^2 & \frac{1}{405}h^2 \\
\frac{22}{135}h^2 & -\frac{2}{135}h^2 & \frac{2}{405}h^2 \\
\frac{3}{10}h^2 & \frac{3}{40}h^2 & \frac{1}{60}h^2 \\
\frac{19}{72}h & \frac{5}{72}h & \frac{1}{72}h \\
\frac{4}{9}h & \frac{1}{9}h & 0 \\
\frac{3}{8}h & \frac{3}{8}h & \frac{1}{8}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1}
\end{bmatrix}
\tag{3.3.25}
$$

Equationn (3.3.25) can be written explicitly as

$$
\begin{aligned}
y_{n+\frac{1}{3}} &= y_n + \frac{1}{3}hy'_n + \frac{h^2}{3240}\left[8f_{n+1} - 39f_{n+\frac{2}{3}} + 114f_{n+\frac{2}{3}} + 97f_n\right] \\
y_{n+\frac{2}{3}} &= y_n + \frac{2}{3}hy'_n + \frac{h^2}{405}\left[2f_{n+1} - 6f_{n+\frac{2}{3}} + 66f_{n+\frac{1}{3}} + 28f_n\right] \\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{120}\left[2f_{n+1} + 9f_{n+\frac{2}{3}} + 36f_{n+\frac{1}{3}} + 13f_n\right] \\
y'_{n+\frac{1}{3}} &= y'_n + \frac{h}{72}\left[f_{n+1} - 5f_{n+\frac{2}{3}} + 19f_{n+\frac{1}{3}} + 9f_n\right] \\
y'_{n+\frac{2}{3}} &= y'_n + \frac{h}{9}\left[f_{n+\frac{2}{3}} + 4f_{n+\frac{1}{3}} + f_n\right] \\
y'_{n+1} &= y'_n + \frac{h}{8}\left[f_{n+1} + 3f_{n+\frac{2}{3}} + 3f_{n+\frac{1}{3}} + f_n\right]
\end{aligned}
\tag{3.3.26}
$$

3.3.3 One Step Method with Three Offstep Points

Here, three offstep points have been introduced. Similarly, these points are carefully chosen to guarantee the zero stability of the method. Here, $i = 1, 2, 3$, which implies that we have $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{2}$ and $\nu_3 = \frac{3}{4}$.

Thus, from (3.2.1) for $r = 5$ and $s = 2$, we obtain a polynomial of degree $r + s - 1$ of the form

$$y(x) = \sum_{j=0}^{6} a_j x^j \tag{3.3.27}$$

with second derivative given by

$$y''(x) = \sum_{j=0}^{6} j(j-1) a_j x^{j-2} \tag{3.3.28}$$

Substituting (3.3.28) into (1.8.2) gives

$$\sum_{j=0}^{6} j(j-1) a_j x^{j-2} = f(x, y, y') \tag{3.3.29}$$

Collocating (3.3.29) at $x_{n+r}, r = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and interpolating (3.3.27) at $x_{n+s}, s = \frac{1}{2}$ and $\frac{3}{4}$, leads to a system of equations written in the matrix form $AX = B$ as follows:

$$\begin{bmatrix} 1 & x_{n\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} \\ 1 & x_{n+\frac{3}{4}} & x^2_{n+\frac{3}{4}} & x^3_{n+\frac{3}{4}} & x^4_{n+\frac{3}{4}} & x^5_{n+\frac{3}{4}} & x^6_{n+\frac{3}{4}} \\ 0 & 0 & 2 & 6x_n & 12x^2_n & 20x^3_n & 30x^4_n \\ 0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x^2_{n+\frac{1}{4}} & 20x^3_{n+\frac{1}{4}} & 30x^4_{n+\frac{1}{4}} \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12x^2_{n+\frac{3}{4}} & 20x^3_{n+\frac{3}{4}} & 30x^4_{n+\frac{3}{4}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \tag{3.3.30}$$

Solving (3.3.30) by Gaussian elimination yield the a_j's as follows:

$$\begin{aligned}
a_0 &= y_{n+\frac{1}{2}} - a_1 x_{n+\frac{1}{2}} - a_2 x^2_{n+\frac{1}{2}} - a_3 x^3_{n+\frac{1}{2}} - a_4 x^4_{n+\frac{1}{2}} - a_5 x^5_{n+\frac{1}{2}} - a_6 x^6_{n+\frac{1}{2}} \\
a_1 &= \frac{y_{n+\frac{3}{4}} - y_{n+\frac{1}{2}}}{\frac{1}{4}h} - a_2\left(x_{n+\frac{3}{4}} + x_{n+\frac{1}{2}}\right) - a_3\left(x^2_{n+\frac{3}{4}} + x_{n+\frac{3}{4}}x_{n+\frac{1}{2}} + x^2_{n+\frac{1}{1}}\right) \\
&\quad -a_4\left(x^3_{n+\frac{3}{4}} + x^2_{n+\frac{3}{4}}x_{n+\frac{1}{2}} + x_{n+\frac{3}{4}}x^2_{n+\frac{1}{2}} + x^3_{n+\frac{1}{2}}\right) - a_5\left(x^4_{n+\frac{3}{4}} + x^3_{n+\frac{3}{4}}x_{n+\frac{1}{2}}\right. \\
&\quad \left.+x^2_{n+\frac{3}{4}}x^2_{n+\frac{1}{2}} + x_{n+\frac{3}{4}}x^3_{n+\frac{1}{2}} + x^4_{n+\frac{1}{2}}\right) - a_6\left(x^5_{n+\frac{3}{4}} + x^4_{n+\frac{3}{4}}x_{n+\frac{1}{2}} + x^3_{n+\frac{3}{4}}x^2_{n+\frac{1}{2}}\right. \\
&\quad \left.+x^2_{n+\frac{3}{4}}x^3_{n+\frac{1}{2}} + x_{n+\frac{3}{4}}x^4_{n+\frac{1}{2}} + x^5_{n+\frac{1}{2}}\right) \\
a_2 &= \frac{1}{2}f_n - 3a_3x_n - 6a_4x^2_n - 10x^3_n - 15x^4_n \\
a_3 &= \frac{f_{n+\frac{1}{4}} - f_n}{\frac{3}{2}h} - 2a_4\left(x_{n+\frac{1}{4}} + x_n\right) + \frac{10}{3}a_5\left(x^2_{n+\frac{1}{4}} + x_{n+\frac{1}{4}}x_n + x^2_n\right) \\
&\quad -5a_6\left(x^3_{n+\frac{1}{4}} + x^2_{n+\frac{1}{4}}x_n + x_{n+\frac{1}{4}}x^2_n + x^3_n\right) \qquad (3.3.31) \\
a_4 &= \frac{\frac{1}{4}f_{n+\frac{1}{2}} - \frac{1}{2}f_{n+\frac{1}{4}} + \frac{1}{4}f_n}{\frac{3}{8}h^2} - \frac{5}{3}a_5\left(x_{n+\frac{1}{2}} + x_{n+\frac{1}{4}} + x_n\right) \\
&\quad -\frac{5}{2}a_6\left(x^2_{n+\frac{1}{2}} + x_{n+\frac{1}{2}}x_{n+\frac{1}{4}} + x^2_{n+\frac{1}{4}} + x_{n+\frac{1}{2}}x_n + x_{n+\frac{1}{4}}x_n + x^2_n\right) \\
a_5 &= \frac{\frac{1}{32}f_{n+\frac{3}{4}} - \frac{3}{32}f_{n+\frac{1}{2}} + \frac{3}{32}f_{n+\frac{1}{4}} - \frac{1}{32}f_n}{\frac{15}{256}h^3} - \frac{3}{2}a_6\left(x_{n+\frac{3}{4}} + x_{n+\frac{1}{2}} + x_{n+\frac{1}{4}} + x_n\right) \\
a_6 &= \frac{\frac{3}{1024}f_{n+1} - \frac{3}{256}f_{n+\frac{3}{4}} + \frac{9}{512}f_{n+\frac{1}{2}} - \frac{3}{256}f_{n+\frac{1}{4}} + \frac{3}{1024}f_n}{\frac{135}{16384}h^4}
\end{aligned}$$

Substituting (3.3.31) into (3.3.27) yields the continuous implicit hybrid one step method in the form of a continuous linear multistep method described by the formula

$$y(x) = \sum_{j=0}^{k}\alpha_j(x)y_{n+j} + \sum_{\nu_i}\alpha_{\nu_i}(x)y_{n+\nu_i} + h^2\left[\sum_{j=0}^{k}\beta_j(x)f_{n+j} + \sum_{\nu_i}\beta_{\nu_i}(x)f_{n+\nu_i}\right], \quad i = 1, 2, 3 \qquad (3.3.32)$$

For $k = 1$, $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{2}$ and $\nu_3 = \frac{3}{4}$ and writing $\alpha_j(x)$ and $\beta_j(x)$, $\quad j = 0, 1, 2$, as continuous functions of t, where $t = \frac{x - x_{n+\frac{3}{4}}}{h}$, we obtain the parameters

$$
\begin{aligned}
\alpha_{\frac{1}{2}}(t) &= -4t \\
\alpha_{\frac{3}{4}}(t) &= 4t + 1 \\
\beta_0(t) &= \frac{h^2}{5760}(2048t^6 + 1536t^5 - 320t^4 - 320t^3 + 11t) \\
\beta_{\frac{1}{4}}(t) &= -\frac{h^2}{720}(1024t^6 + 1152t^5 - 160t^4 - 240t^3 + 9t) \\
\beta_{\frac{1}{2}}(t) &= \frac{h^2}{960}(2048t^6 + 3072t^5 + 320t^4 - 960t^3 + 55t) \\
\beta_{\frac{3}{4}}(t) &= -\frac{h^2}{720}(1024t^6 + 1320t^5 + 800t^4 - 400t^3 - 360t^2 - 59t) \\
\beta_1(t) &= \frac{h^2}{5760}(2048t^6 + 4608t^5 + 7920t^4 + 960t^3 - 21t)
\end{aligned}
\tag{3.3.33}
$$

Evaluating (3.3.32) at $x_n, x_{n+\frac{1}{4}}$ and x_{n+1} implies in (3.2.8) that $t = -\frac{3}{4}$, $-\frac{1}{2}$ and $\frac{1}{4}$ respectively, which yields the following discrete methods

$$2y_{n+\frac{3}{4}} - 3y_{n+\frac{1}{2}} + y_n = \frac{h^2}{3840}\left[-3f_{n+1} + 52f_{n+\frac{3}{4}} + 402f_{n+\frac{1}{2}} + 252f_{n+\frac{1}{4}} + 17f_n\right] \quad (3.3.34a)$$

$$y_{n+\frac{3}{4}} - 2y_{n+\frac{1}{2}} + y_{n+\frac{1}{4}} = -\frac{h^2}{3840}\left[f_{n+1} - 24f_{n+\frac{3}{4}} - 194f_{n+\frac{1}{2}} - 24f_{n+\frac{1}{4}} + f_n\right] \quad (3.3.34b)$$

$$y_{n+1} - 2y_{n+\frac{3}{4}} + y_{n+\frac{1}{2}} = \frac{h^2}{3840}\left[19f_{n+1} + 204f_{n+\frac{3}{4}} - 14f_{n+\frac{1}{2}} + 4f_{n+\frac{1}{4}} - f_n\right] \quad (3.3.34c)$$

Differentiating (3.3.33) gives

$$\begin{aligned}
\alpha'_{\frac{1}{2}}(t) &= \frac{-4}{h} \\
\alpha'_{\frac{3}{4}}(t) &= \frac{4}{h} \\
\beta'_{0}(t) &= \frac{h}{5760}(12288t^5 + 7680t^4 - 1280t^3 - 960t^2 + 11) \\
\beta'_{\frac{1}{4}}(t) &= -\frac{h}{720}(6144t^5 - 5760t^4 - 640t^3 - 720t^2 + 9) \\
\beta'_{\frac{1}{2}}(t) &= \frac{h}{960}(12288t^5 + 15360t^4 + 1280t^3 - 2880t^2 + 55) \\
\beta'_{\frac{3}{4}}(t) &= -\frac{h}{720}(6144t^5 + 11520t^4 + 3200t^3 - 1200t^2 - 720t - 59) \\
\beta'_{1}(t) &= \frac{h}{5760}(12288t^5 + 23040t^4 + 14080t^3 + 2880t^2 - 21)
\end{aligned} \tag{3.3.35}$$

Evaluating (3.3.35) for $t = -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}$, respectively, yields the following discrete derivative methods

$$\begin{aligned}
hy'_n - 4y_{n+\frac{3}{4}} + 4y_{n+\frac{1}{2}} &= \frac{h^2}{5760}\left[33f_{n+1} - 284f_{n+\frac{3}{4}} - 966f_{n+\frac{1}{2}} - 1908f_{n+\frac{1}{4}} - 475f_n\right] \\
hy'_{n+\frac{1}{4}} - 4y_{n+\frac{3}{4}} + 4y_{n+\frac{1}{2}} &= \frac{h^2}{5760}\left[-5f_{n+1} - 72f_{n+\frac{3}{4}} - 1494f_{n+\frac{1}{2}} - 616f_{n+\frac{1}{4}} + 27f_n\right] \\
hy'_{n+\frac{1}{2}} - 4y_{n+\frac{3}{4}} + 4y_{n+\frac{1}{2}} &= \frac{h^2}{5760}\left[17f_{n+1} - 220f_{n+\frac{3}{4}} - 582f_{n+\frac{1}{2}} + 76f_{n+\frac{1}{4}} - 11f_n\right] \\
hy'_{n+\frac{3}{4}} - 4y_{n+\frac{3}{4}} + 4y_{n+\frac{1}{2}} &= \frac{h^2}{5760}\left[-21f_{n+1} + 472f_{n+\frac{3}{4}} + 330f_{n+\frac{1}{2}} - 72f_{n+\frac{1}{4}} + 11f_n\right] \\
hy'_{n+1} - 4y_{n+\frac{3}{4}} + 4y_{n+\frac{1}{2}} &= \frac{h^2}{5760}\left[481f_{n+1} + 1764f_{n+\frac{3}{4}} - 198f_{n+\frac{1}{2}} + 140f_{n+\frac{1}{4}} - 27f_n\right]
\end{aligned} \tag{3.3.36}$$

Combining (3.3.34) and (3.3.36), and using the modified block formulae (3.2.11) and (3.2.12) respectively, we have

$$\begin{bmatrix} 0 & -11520 & 7680 & 0 & 0 & 0 & 0 & 0 \\ 3840 & -7680 & 3840 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3840 & -7680 & 3840 & 0 & 0 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 0 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 5760h0 & 0 & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 5760h & 0 & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 0 & 5760h & 0 \\ 0 & 23040 & -23040 & 0 & 0 & 0 & 0 & 5760h \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{3}{4}} \\ y'_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} -3840 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -5760h \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \begin{bmatrix} 17h^2 \\ -h^2 \\ -h^2 \\ -475h^2 \\ 27h^2 \\ -11h^2 \\ 11h^2 \\ -27h^2 \end{bmatrix} [f_n]$$

$$
+\begin{bmatrix} 252h^2 & 402h^2 & 52h^2 & -3h^2 \\ 24h^2 & 194h^2 & 24h^2 & -h^2 \\ 4h^2 & 204h^2 & 204h^2 & 19h^2 \\ 1908h^2 & -966h^2 & -284h^2 & 33h^2 \\ -616h^2 & -1494h^2 & -72h^2 & -5h^2 \\ 76h^2 & -582h^2 & -220h^2 & 17h^2 \\ -72h^2 & 330h^2 & 472h^2 & -21h^2 \\ 140h^2 & -198h^2 & 1764h^2 & 481h^2 \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix} \tag{3.3.37}
$$

and

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{3}{4}} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4}h \\ 1 & \frac{1}{2}h \\ 1 & \frac{3}{4}h \\ 1 & h \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \begin{bmatrix} \frac{367}{23040}h^2 \\ \frac{53}{1440}h^2 \\ \frac{147}{2560}h^2 \\ \frac{7}{90}h^2 \\ \frac{251}{2880}h \\ \frac{29}{360}h \\ \frac{27}{320}h \\ \frac{7}{90}h \end{bmatrix} [f_n]
$$

$$
+\begin{bmatrix}
\frac{3}{128}h^2 & -\frac{47}{3840}h^2 & \frac{29}{5760}h^2 & -\frac{7}{7880}h^2 \\
\frac{1}{10}h^2 & -\frac{1}{48}h^2 & \frac{1}{90}h^2 & -\frac{1}{480}h^2 \\
\frac{117}{640}h^2 & \frac{27}{1280}h^2 & \frac{3}{128}h^2 & -\frac{9}{2560}h^2 \\
\frac{4}{15}h^2 & \frac{1}{15}h^2 & \frac{4}{45}h^2 & 0 \\
\frac{325}{1440}h & -\frac{11}{120}h & \frac{53}{1440}h & -\frac{19}{2880}h \\
\frac{31}{90}h & \frac{1}{5}h & \frac{1}{90}h & -\frac{1}{360}h \\
\frac{51}{160}h & \frac{9}{40}h & \frac{21}{160}h & -\frac{3}{320}h \\
\frac{16}{45}h & \frac{2}{15}h & \frac{16}{45}h & \frac{7}{90}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{4}} \\
f_{n+\frac{1}{2}} \\
f_{n+\frac{3}{4}} \\
f_{n+1}
\end{bmatrix}
\tag{3.3.38}
$$

Equation (3.3.38) is written explicitly as

$$
\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + \frac{h^2}{23040}\left[-21f_{n+1} + 116f_{n+\frac{3}{4}} - 282f_{n+\frac{1}{2}} + 540f_{n+\frac{1}{4}} + 367f_n\right] \\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + \frac{h^2}{1440}\left[-3f_{n+1} + 16f_{n+\frac{3}{4}} - 30f_{n+\frac{1}{2}} + 144f_{n+\frac{1}{4}} + 53f_n\right] \\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + \frac{h^2}{2560}\left[-3f_{n+1} + 20f_{n+\frac{3}{4}} + 18f_{n+\frac{1}{2}} + 156f_{n+\frac{1}{4}} + 49f_n\right] \\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{90}\left[8f_{n+\frac{3}{4}} + 6f_{n+\frac{1}{2}} + 24f_{n+\frac{1}{4}} + 7f_n\right] \qquad (3.3.39) \\
y'_{n+\frac{1}{4}} &= y'_n + \frac{h}{2880}\left[-19f_{n+1} + 106f_{n+\frac{3}{4}} - 264f_{n+\frac{1}{2}} + 646f_{n+\frac{1}{4}} + 251f_n\right] \\
y'_{n+\frac{1}{2}} &= y'_n + \frac{h}{360}\left[-f_{n+1} + 4f_{n+\frac{3}{4}} + 24f_{n+\frac{1}{2}} + 124f_{n+\frac{1}{4}} + 29f_n\right] \\
y'_{n+\frac{3}{4}} &= y'_n + \frac{h}{320}\left[-3f_{n+1} + 42f_{n+\frac{3}{4}} + 72f_{n+\frac{1}{2}} + 102f_{n+\frac{1}{4}} + 27f_n\right] \\
y'_{n+1} &= y'_n + \frac{h}{90}\left[7f_{n+1} + 32f_{n+\frac{3}{4}} + 12f_{n+\frac{1}{2}} + 32f_{n+\frac{1}{4}} + 7f_n\right]
\end{aligned}
$$

3.3.4 One Step Method with Four Offstep Points

Four offstep points have been introduced to derive this method. Like the previous cases, the points are carefully chosen to guarantee the zero stability of the method. Here, $i = 1, 2, 3, 4$, thus we have $\nu_1 = \frac{1}{5}$, $\nu_2 = \frac{2}{5}$, and $\nu_3 = \frac{3}{5}$ and $\nu_4 = \frac{4}{5}$.

Thus, from (3.2.1) for $r = 6$ and $s = 2$ we obtain the basis polynomial of degree $r + s - 1$ of the form

$$y(x) = \sum_{j=0}^{7} a_j x^j \tag{3.3.40}$$

with second derivative given by

$$y'' = \sum_{j=0}^{7} j(j-1)a_j x^{j-2} \tag{3.3.41}$$

Substituting (3.3.41) into (1.8.2) gives the differential system

$$\sum_{j=0}^{7} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3.3.42}$$

Collocating (3.3.42) at $x_{n+r}, r = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$ and interpolating (3.3.40) at $x_{n+s}, s = \frac{3}{5}$ and $\frac{4}{5}$ leads to a system of equations written in the matrix form $AX = B$ as follows

$$\begin{bmatrix} 1 & x_{n+\frac{3}{5}} & x^2_{n+\frac{3}{5}} & x^3_{n+\frac{3}{5}} & x^4_{n+\frac{3}{5}} & x^5_{n+\frac{3}{5}} & x^6_{n+\frac{3}{5}} & x^7_{n+\frac{3}{5}} \\ 1 & x_{n+\frac{4}{5}} & x^2_{n+\frac{4}{5}} & x^3_{n+\frac{4}{5}} & x^4_{n+\frac{4}{5}} & x^5_{n+\frac{4}{5}} & x^6_{n+\frac{4}{5}} & x^7_{n+\frac{4}{5}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{5}} & 12x^2_{n+\frac{1}{5}} & 20x^3_{n+\frac{1}{5}} & 30x^4_{n+\frac{1}{5}} & 42x^5_{n+\frac{1}{5}} \\ 0 & 0 & 2 & 6x_{n+\frac{2}{5}} & 12x^2_{n+\frac{2}{5}} & 20x^3_{n+\frac{2}{5}} & 30x^4_{n+\frac{2}{5}} & 42x^5_{n+\frac{2}{5}} \\ 0 & 0 & 2 & 6x_{n+\frac{3}{5}} & 12x^2_{n+\frac{3}{5}} & 20x^3_{n+\frac{3}{5}} & 30x^4_{n+\frac{3}{5}} & 42x^5_{n+\frac{3}{5}} \\ 0 & 0 & 2 & 6x_{n+\frac{4}{5}} & 12x^2_{n+\frac{4}{5}} & 20x^3_{n+\frac{4}{5}} & 30x^4_{n+\frac{4}{5}} & 42x^5_{n+\frac{4}{5}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^4_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} \tag{3.3.43}$$

Solving (3.3.43) by Gaussian elimination method, the a_j's are obtained as follows

$$\begin{aligned}
a_0 &= y_{n+\frac{3}{5}} - a_1 x_{n+\frac{3}{5}} - a_2 x^2_{n+\frac{3}{5}} - a_3 x^3_{n+\frac{3}{5}} - a_4 x^4_{n+\frac{3}{5}} - a_5 x^5_{n+\frac{3}{5}} - a_6 x^6_{n+\frac{3}{5}} - a_7 x^7_{n+\frac{3}{5}} \\
a_1 &= \frac{y_{n+\frac{4}{5}} - y_{n+\frac{3}{5}}}{\frac{1}{5}h} - a_2\left(x_{n+\frac{4}{5}} + x_{n+\frac{3}{5}}\right) - a_3\left(x^2_{n+\frac{4}{5}} + x_{n+\frac{4}{5}}x_{n+\frac{3}{5}} + x^2_{n+\frac{3}{5}}\right) \\
&\quad -a_4\left(x^3_{n+\frac{4}{5}} + x^2_{n+\frac{4}{5}}x_{n+\frac{3}{5}} + x_{n+\frac{4}{5}}x^2_{n+\frac{3}{5}} + x^3_{n+\frac{3}{5}}\right) - a_5\left(x^4_{n+\frac{4}{5}} + x^3_{n+\frac{4}{5}}x_{n+\frac{4}{5}} + x^2_{n+\frac{4}{5}}x^2_{n+\frac{3}{5}}\right. \\
&\quad \left. . + x_{n+\frac{4}{5}}x^3_{n+\frac{3}{5}} + x^4_{n+\frac{3}{5}}\right) - a_6\left(x^5_{n+\frac{4}{5}} + x^4_{n+\frac{4}{5}}x_{n+\frac{3}{5}} + x^3_{n+\frac{4}{5}}x^2_{n+\frac{3}{5}} + x^2_{n+\frac{4}{5}}x^3_{n+\frac{3}{5}}\right. \\
&\quad \left. +x_{n+\frac{4}{5}}x^4_{n+\frac{3}{5}} + x^5_{n+\frac{3}{5}}\right) - a_7\left(x^6_{n+\frac{4}{5}} + x^5_{n+\frac{4}{5}}x_{n+\frac{3}{5}} + x^4_{n+\frac{4}{5}}x^2_{n+\frac{3}{5}}\right. \\
&\quad \left. +x^3_{n+\frac{4}{5}}x^3_{n+\frac{3}{5}} + x^2_{n+\frac{4}{5}}x^4_{n+\frac{3}{5}} + x_{n+\frac{4}{5}}x^5_{n+\frac{3}{5}} + x^6_{n+\frac{3}{5}}\right) \\
a_2 &= \frac{1}{2}f_n - 3a_3x_n - 6a_4x_n^2 - 10a_5x_n^3 - 15a_6x_n^4 - 21a_7x_n^5 \\
a_3 &= \frac{f_{n+\frac{1}{5}} - f_n}{\frac{6}{5}h} - 2a_4\left(x_{n+\frac{1}{5}} + x_n\right) - \frac{10}{3}a_5\left(x^2_{n+\frac{1}{5}} + x_{n+\frac{1}{5}}x_n + x_n^2\right) \\
&\quad -5a_6\left(x^3_{n+\frac{1}{5}} + x^2_{n+\frac{1}{5}}x_n + x_{n+\frac{1}{5}}x_n^2 + x_n^3\right) - 7a_7\left(x^4_{n+\frac{1}{5}} + x^3_{n+\frac{1}{5}}x_n + x^2_{n+\frac{1}{5}} + x_{n+\frac{1}{5}}x_n^3 + x_n^4\right) \\
a_4 &= \frac{\frac{1}{5}f_{n+\frac{2}{5}} - \frac{2}{5}f_{n+\frac{1}{5}} + \frac{1}{5}f_n}{\frac{24}{125}h^2} - \frac{5}{3}a_5\left(x_{n+\frac{2}{5}} + x_{n+\frac{1}{5}} + x_n\right) \\
&\quad -\frac{5}{2}a_6\left(x^2_{n+\frac{2}{5}} + x_{n+\frac{2}{5}}x_{n+\frac{1}{5}} + x^2_{n+\frac{1}{5}} + x_{n+\frac{2}{5}}x_n + x_{n+\frac{1}{5}}x_n + x_n^2\right) \\
&\quad -\frac{7}{2}a_7\left(x^3_{n+\frac{2}{5}} + x^2_{n+\frac{2}{5}}x_{n+\frac{1}{5}} + x_{n+\frac{2}{5}}x^2_{n+\frac{1}{5}} + x^3_{n+\frac{1}{5}} + x^2_{n+\frac{2}{5}}x_n + x_{n+\frac{2}{5}}x_{n+\frac{1}{5}}x_n\right. \\
&\quad \left. +x^2_{n+\frac{1}{5}}x_n + x_{n+\frac{2}{5}}x_n^2 + x_{n+\frac{1}{5}}x_n^2 + x_n^3\right) \qquad (3.3.44) \\
a_5 &= \frac{\frac{2}{125}f_{n+\frac{3}{5}} - \frac{6}{125}f_{n+\frac{2}{5}} + \frac{6}{125}f_{n+\frac{1}{5}} - \frac{2}{125}f_n}{\frac{48}{3125}h^3} - \frac{3}{2}a_6\left(x_{n+\frac{3}{5}} + x_{n+\frac{2}{5}} + x_{n+\frac{1}{5}} + x_n\right) \\
&\quad -\frac{21}{10}a_7\left(x^2_{n+\frac{3}{5}} + x_{n+\frac{3}{5}}x_{n+\frac{2}{5}} + x^2_{n+\frac{2}{5}} + x_{n+\frac{3}{5}}x_{n+\frac{1}{5}} + x_{n+\frac{2}{5}}x_{n+\frac{1}{5}} + x^2_{n+\frac{1}{5}} + x_{n+\frac{3}{5}}x_n\right. \\
&\quad \left. +x_{n+\frac{2}{5}}x_n + x_{n+\frac{1}{5}}x_n + x_n^2\right) \\
a_6 &= \frac{\frac{12}{15625}f_{n+\frac{4}{5}} - \frac{48}{15625}f_{n+\frac{3}{5}} + \frac{72}{15625}f_{n+\frac{2}{5}} - \frac{48}{15625}f_{n+\frac{1}{5}} + \frac{12}{15625}f_n}{\frac{1728}{1943125}h^4} \\
&\quad -\frac{7}{5}a_7\left(x_{n+\frac{4}{5}} + x_{n+\frac{3}{5}} + x_{n+\frac{2}{5}} + x_{n+\frac{1}{5}} + x_n\right) \\
a_7 &= \frac{\frac{288}{9765625}f_{n+1} - \frac{288}{1953125}f_{n+\frac{4}{5}} + \frac{576}{1953125}f_{n+\frac{3}{5}} - \frac{576}{1953125}f_{n+\frac{2}{5}} + \frac{288}{1953125}f_{n+\frac{1}{5}} - \frac{288}{9765625}f_n}{\frac{290304}{6103515625}h^5}
\end{aligned}$$

Substituting (3.3.44) into (3.3.40) with some manipulation leads to the continuous implicit hybrid one step method in the form of a CLMMs described by the formula

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{\nu_i} \alpha_{\nu_i}(x) y_{n+\nu_i} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \sum_{\nu_i} \beta_{\nu_i}(x) f_{n+\nu_i} \right], \quad i = 1, 2, 3, 4 \quad (3.3.45)$$

For $k = 1$, $\nu_1 = \frac{1}{5}, \nu_2 = \frac{2}{5}, \nu_3 = \frac{3}{5}, \nu_4 = \frac{4}{5}$ and writing $\alpha_j(x)$ and $\beta_j(x)$, $j = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$, as continuous functions of t, where $t = \frac{x - x_{n+\frac{4}{5}}}{h}$, we obtain the following parameters

$$\begin{aligned}
\alpha_{\frac{3}{5}}(t) &= -5t \\
\alpha_{\frac{4}{5}}(t) &= 5t + 1 \\
\beta_0(t) &= \frac{-h^2}{10080}(6250t^7 + 875t^6 + 2625t^5 - 875t^4 - 420t^3 + 8t) \\
\beta_{\frac{1}{5}}(t) &= \frac{h^2}{50400}(156250t^7 + 262500t^6 + 91875t^5 - 26250t^4 - 14000t^3 + 277t) \\
\beta_{\frac{2}{5}}(t) &= \frac{-h^2}{25200}(156250t^7 + 306250t^6 + 144375t^5 - 30625t^4 - 21000t^3 + 452t) \\
\beta_{\frac{3}{5}}(t) &= \frac{h^2}{5040}(31250t^7 + 70000t^6 + 44625t^5 - 1750t^4 - 8400t^3 + 271t) \\
\beta_{\frac{4}{5}}(t) &= \frac{-h^2}{50400}(156250t^7 + 393750t^6 + 328125t^5 + 65625t^4 - 45500t^3 - 25200t^2 - 3104t) \\
\beta_1(t) &= \frac{h^2}{50400}(31250t^7 + 87500t^6 + 91875t^5 + 43750t^4 + 8400t^3 - 107t)
\end{aligned} \quad (3.3.46)$$

Evaluating (3.3.45) at $x_n, x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}$ and x_{n+1} and using (3.2.8) gives $t = -\frac{4}{5}$, $-\frac{3}{5}$, $-\frac{2}{5}, \frac{1}{5}$. Thus, from (3.3.46), the following discrete methods are obtained;

$$3y_{n+\frac{4}{5}} - 4y_{n+\frac{3}{5}} + y_n = \frac{h^2}{3000}\left[-f_{n+1} + 32f_{n+\frac{4}{5}} + 322f_{n+\frac{3}{5}} + 232f_{n+\frac{2}{5}} + 127f_{n+\frac{1}{5}} + 8f_n\right] \quad (3.3.47a)$$

$$2y_{n+\frac{4}{5}} - 3y_{n+\frac{3}{5}} + y_{n+\frac{1}{5}} = \frac{-h^2}{6000}\left[2f_{n+1} - 47f_{n+\frac{4}{5}} - 412f_{n+\frac{3}{5}} - 242f_{n+\frac{2}{5}} - 22f_{n+\frac{1}{5}} + f_n\right] \quad (3.3.47b)$$

$$y_{n+\frac{4}{5}} - 2y_{n+\frac{3}{5}} + y_{n+\frac{2}{5}} = \frac{-h^2}{6000}\left[f_{n+1} - 24f_{n+\frac{4}{5}} - 194f_{n+\frac{3}{5}} - 24f_{n+\frac{2}{5}} + f_{n+\frac{1}{5}}\right] \quad (3.3.47c)$$

$$y_{n+1} - 2y_{n+\frac{4}{5}} + y_{n+\frac{3}{5}} = \frac{h^2}{6000}\left[18f_{n+1} + 209f_{n+\frac{4}{5}} + 4f_{n+\frac{3}{5}} + 14f_{n+\frac{2}{5}} - 6f_{n+\frac{1}{5}} + f_n\right] \quad (3.3.47d)$$

Differentiating (3.3.46), gives the following:

$$\begin{aligned}
\alpha'_{\frac{3}{5}}(t) &= -\frac{5}{h} \\
\alpha'_{\frac{4}{5}}(t) &= \frac{5}{h} \\
\beta'_{0}(t) &= -\frac{h}{10080}(43750t^6 + 52500t^5 + 13125t^4 - 3500t^3 - 1260t^2 + 8) \\
\beta'_{\frac{1}{5}}(t) &= \frac{h}{50400}(1093750t^6 + 1575000t^5 + 459375t^4 - 105000t^3 - 42000t^2 + 277) \\
\beta'_{\frac{2}{5}}(t) &= -\frac{h}{25200}(1093750t^6 + 1837500t^5 + 721875t^4 - 122500t^3 - 63000t^2 + 452) \\
\beta'_{\frac{3}{5}}(t) &= \frac{h}{5040}(218750t^6 + 420000t^5 + 223125t^4 - 7000t^3 - 25200t^2 + 271) \\
\beta'_{\frac{4}{5}}(t) &= -\frac{h}{50400}(1093750t^6 + 2362500t^5 + 1640625t^4 + 262500t^3 - 136500t^2 - 50400t - 3104) \\
\beta'_{1}(t) &= \frac{h}{50400}(218750t^6 + 525000t^5 + 459375t^4 + 175000t^3 + 25200t^2 - 107)
\end{aligned} \tag{3.3.48}$$

Evaluating (3.3.48) at $x_n, x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}, x_{n+\frac{4}{5}}$ and x_{n+1} using (3.2.8) gives $t = -\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}, 0, \frac{1}{5}$. Thus, the following discrete derivative methods are obtained;

$$
\begin{aligned}
hy'_n - 5y_{n+\frac{4}{5}} + 5y_{n+\frac{3}{5}} &= -\frac{h}{50400}\Big[107f_{n+1} + 32f_{n+\frac{4}{5}} + 11626f_{n+\frac{3}{5}} + 6280f_{n+\frac{2}{5}} \\
&\qquad +14059f_{n+\frac{1}{5}} + 3176f_n\Big] \\
hy'_{n+\frac{1}{5}} - 5y_{n+\frac{4}{5}} + 5y_{n+\frac{3}{5}} &= \frac{h}{50400}\Big[82f_{n+1} - 1243f_{n+\frac{4}{5}} - 8252f_{n+\frac{3}{5}} - 11866f_{n+\frac{2}{5}} \\
&\qquad -4070f_{n+\frac{1}{5}} + 149f_n\Big] \\
hy'_{n+\frac{2}{5}} - 5y_{n+\frac{4}{5}} + 5y_{n+\frac{3}{5}} &= \frac{-h}{50400}\Big[-5f_{n+1} + 704f_{n+\frac{4}{5}} + 10058f_{n+\frac{3}{5}} + 4712f_{n+\frac{2}{5}} \\
&\qquad -389f_{n+\frac{1}{5}} + 40f_n\Big] \\
hy'_{n+\frac{3}{5}} - 5y_{n+\frac{4}{5}} + 5y_{n+\frac{3}{5}} &= \frac{h}{50400}\Big[82f_{n+1} - 1355f_{n+\frac{4}{5}} - 4444f_{n+\frac{3}{5}} + 902f_{n+\frac{2}{5}} \\
&\qquad -262f_{n+\frac{1}{5}} + 37f_n\Big] \\
hy'_{n+\frac{4}{5}} - 5y_{n+\frac{4}{5}} + 5y_{n+\frac{3}{5}} &= -\frac{h}{50400}\Big[107f_{n+1} - 3104f_{n+\frac{4}{5}} - 2710f_{n+\frac{3}{5}} + 904f_{n+\frac{2}{5}} \\
&\qquad -277f_{n+\frac{1}{5}} + 40f_n\Big] \\
hy'_{n+1} - 5y_{n+\frac{4}{5}} + 5y_{n+\frac{3}{5}} &= \tfrac{h}{50400}\Big[3218f_{n+1} + 13093f_{n+\frac{4}{5}} - 2876f_{n+\frac{3}{5}} + 2470f_{n+\frac{2}{5}} \\
&\qquad -934f_{n+\frac{1}{5}} + 149f_n\Big]
\end{aligned}
\tag{3.3.49}
$$

Combining (3.3.47) and (3.3.49) using (3.2.11) and (3.2.12) respectively gives

$$\begin{bmatrix} 0 & 0 & -12000 & 9000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6000 & 0 & -18000 & 12000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6000 & -12000 & 6000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6000 & -12000 & 6000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 252000 & -252000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 252000 & -252000 & 0 & 50400h & 0 & 0 & 0 & 0 \\ 0 & 0 & 252000 & -252000 & 0 & 0 & 50400h & 0 & 0 & 0 \\ 0 & 0 & 252000 & -25200 & 0 & 0 & 0 & 50400h & 0 & 0 \\ 0 & 0 & 25200 & -252000 & 0 & 0 & 0 & 0 & 50400h & 0 \\ 0 & 0 & 252000 & -252000 & 0 & 0 & 0 & 0 & 0 & 50400h \end{bmatrix}$$

$$\begin{bmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \\ y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} -3000 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -50400h \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \begin{bmatrix} 8h^2 \\ -h^2 \\ 0 \\ h^2 \\ -3176h^2 \\ 149h^2 \\ -40h^2 \\ 37h^2 \\ -40h^2 \\ 149h^2 \end{bmatrix} [f_n]$$

$$
+\begin{bmatrix}
127h^2 & 232h^2 & 322h^2 & 32h^2 & -h^2 \\
22h^2 & 242h^2 & 412h^2 & 47h^2 & -2h^2 \\
24h^2 & 24h^2 & 194h^2 & 24h^2 & -h^2 \\
-6h^2 & 14h^2 & 4h^2 & 209h^2 & 18h^2 \\
-14059h^2 & -6280h^2 & -11626h^2 & -32h^2 & -107h^2 \\
-4070h^2 & -11866h^2 & -8252h^2 & -1243h^2 & 82h^2 \\
389h^2 & -4712h^2 & -10058h^2 & -704h^2 & 5h^2 \\
-262h^2 & 902h^2 & -4444h^2 & -1355h^2 & 82h^2 \\
277h^2 & -904h^2 & 2710h^2 & 3104h^2 & -107h^2 \\
-934h^2 & 2470h^2 & -2876h^2 & 13093h^2 & 3218h^2
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1}
\end{bmatrix} \tag{3.3.50}
$$

and simplify to obtain

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \\ y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
1 & \frac{1}{5}h \\
1 & \frac{2}{5}h \\
1 & \frac{3}{5}h \\
1 & \frac{4}{5}h \\
1 & h \\
0 & 1 \\
0 & 1 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
y_n \\ y'_n
\end{bmatrix}
+
\begin{bmatrix}
\frac{1231}{12600} \\ \frac{71}{3150} \\ \frac{123}{3500} \\ \frac{376}{7875} \\ \frac{61}{1008} \\ \frac{19}{288} \\ \frac{14}{225} \\ \frac{51}{800} \\ \frac{14}{225} \\ \frac{19}{288}
\end{bmatrix}
[f_n]
$$

$$
+\begin{bmatrix}
\frac{863}{50400}h^2 & -\frac{761}{63000}h^2 & \frac{941}{126000}h^2 & -\frac{341}{126000}h^2 & \frac{107}{252000}h^2 \\
\frac{544}{7875}h^2 & -\frac{37}{1575}h^2 & \frac{136}{7875}h^2 & -\frac{100}{15750}h^2 & \frac{8}{7875}h^2 \\
\frac{350}{28000}h^2 & -\frac{9}{3500}h^2 & \frac{87}{2800}h^2 & -\frac{9}{875}h^2 & \frac{9}{5600}h^2 \\
\frac{1424}{7875}h^2 & \frac{176}{7875}h^2 & \frac{608}{7875}h^2 & -\frac{16}{1575}h^2 & \frac{16}{7875}h^2 \\
\frac{475}{2016}h^2 & \frac{25}{504}h^2 & \frac{125}{1008}h^2 & \frac{25}{1008}h^2 & \frac{11}{2016}h^2 \\
\frac{1427}{7200}h & -\frac{133}{1200}h & \frac{241}{3600}h & -\frac{173}{7200}h & \frac{3}{800}h \\
\frac{43}{150}h & \frac{7}{225}h & \frac{7}{225}h & -\frac{1}{75}h & \frac{1}{450}h \\
\frac{219}{800}h & \frac{57}{400}h & \frac{57}{400}h & -\frac{21}{800}h & \frac{3}{800}h \\
\frac{64}{225}h & \frac{8}{75}h & \frac{64}{225}h & \frac{14}{225}h & 0 \\
\frac{25}{96}h & \frac{25}{144}h & \frac{25}{144}h & \frac{25}{96}h & \frac{19}{288}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1}
\end{bmatrix}
\tag{3.3.51}
$$

Equation (3.3.51) is written explicitly as

$$
\begin{aligned}
y_{n+\frac{1}{5}} &= y_n + \frac{1}{5}hy'_n + \frac{h^2}{252000}\Big[107f_{n+1} - 682f_{n+\frac{4}{5}} + 1882f_{n+\frac{3}{5}} - 3044f_{n+\frac{2}{5}} + 4315f_{n+\frac{1}{5}} \\
&\quad +2462f_n\Big] \\
y_{n+\frac{2}{5}} &= y_n + \frac{2}{5}hy'_n + \frac{h^2}{15750}\Big[10f_{n+1} - 101f_{n+\frac{4}{5}} + 272f_{n+\frac{3}{5}} - 370f_{n+\frac{2}{5}} + 1088f_{n+\frac{1}{5}} \\
&\quad +355f_n\Big] \\
y_{n+\frac{3}{5}} &= y_n + \frac{3}{5}hy'_n + \frac{h^2}{28000}\Big[45f_{n+1} - 288f_{n+\frac{4}{5}} + 870f_{n+\frac{3}{5}} - 72f_{n+\frac{2}{5}} \\
&\quad +3501f_{n+\frac{1}{5}} + 984f_n\Big] \\
y_{n+\frac{4}{5}} &= y_n + \frac{4}{5}hy'_n + \frac{h}{7875}\Big[16f_{n+1} - 80f_{n+\frac{4}{5}} + 608f_{n+\frac{3}{5}} + 176f_{n+\frac{2}{5}} + 1424f_{n+\frac{1}{5}} \\
&\quad +376f_n\Big] \\
y_{n+1} &= y_n + hy'_n + \frac{h}{2016}\Big[11f_{n+1} + 50f_{n+\frac{4}{5}} + 250f_{n+\frac{3}{5}} + 100f_{n+\frac{2}{5}} + 475f_{n+\frac{1}{5}} + 122f_n\Big] \\
y'_{n+\frac{1}{5}} &= y'_n + \frac{h}{7200}\Big[27f_{n+1} - 173f_{n+\frac{4}{5}} + 482f_{n+\frac{3}{5}} - 798f_{n+\frac{2}{5}} + 1427f_{n+\frac{1}{5}} + 475f_n\Big]
\end{aligned}
\tag{3.3.52}
$$

$$y'_{n+\frac{2}{5}} = y'_n + \frac{h}{450}\left[f_{n+1} - 6f_{n+\frac{4}{5}} + 14f_{n+\frac{3}{5}} + 14f_{n+\frac{2}{5}} + 129f_{n+\frac{1}{5}} + 28f_n\right]$$

$$y'_{n+\frac{3}{5}} = y'_n + \frac{h}{800}\left[3f_{n+1} - 21f_{n+\frac{4}{5}} + 114f_{n+\frac{3}{5}} + 114f_{n+\frac{2}{5}} + 219f_{n+\frac{1}{5}} + 51f_n\right]$$

$$y'_{n+\frac{4}{5}} = y'_n + \frac{h}{225}\left[14f_{n+\frac{4}{5}} + 64f_{n+\frac{3}{5}} + 24f_{n+\frac{2}{5}} + 64f_{n+\frac{1}{5}} + 14f_n\right]$$

$$y'_{n+1} = y'_n + \frac{h}{288}\left[19f_{n+1} + 75f_{n+\frac{4}{5}} + 50f_{n+\frac{3}{5}} + 50f_{n+\frac{2}{5}} + 75f_{n+\frac{1}{5}} + 19f_n\right]$$

3.3.5 One Step Method with Five Offstep Points

This method is derived by the addition of five offstep points between x_n and x_{n+1}. The offstep points are chosen carefully to guarantee zero stability of the method. In particular, $i = 1, 2, 3, 4, 5$ which implies that $\nu_1 = \frac{1}{6}$, $\nu_2 = \frac{1}{3}$, $\nu_3 = \frac{1}{2}$, $\nu_4 = \frac{2}{3}$ and $\nu_5 = \frac{5}{6}$.

Thus, (3.2.1) for $r = 7$ and $s = 2$ becomes a polynomial of degree $r + s - 1$ of the form

$$y(x) = \sum_{j=0}^{8} a_j x^j \tag{3.3.53}$$

with second derivative given by

$$y'' = \sum_{j=0}^{8} j(j-1)a_j x^{j-2} \tag{3.3.54}$$

Substituting (3.3.54) into (1.8.2) gives the differential system

$$\sum_{j=0}^{8} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{3.3.55}$$

Collocating (3.3.55) at $x_{n+r}, r = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$ and interpolating (3.3.53) at $x_{n+s}, s = \frac{2}{3}, \frac{5}{6}$ leads to a system of equations written in the matrix form $AX = B$ as follows:

$$\begin{bmatrix} 1 & x_{n+\frac{2}{3}} & x^2_{n+\frac{2}{3}} & x^3_{n+\frac{2}{3}} & x^4_{n+\frac{2}{3}} & x^5_{n+\frac{2}{3}} & x^6_{n+\frac{2}{3}} & x^7_{n+\frac{2}{3}} & x^8_{n+\frac{2}{3}} \\ 1 & x_{n+\frac{5}{6}} & x^2_{n+\frac{5}{6}} & x^3_{n+\frac{5}{6}} & x^4_{n+\frac{5}{6}} & x^5_{n+\frac{5}{6}} & x^6_{n+\frac{5}{6}} & x^7_{n+\frac{5}{6}} & x^8_{n+\frac{5}{6}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{6}} & 12x^2_{n+\frac{1}{6}} & 20x^3_{n+\frac{1}{6}} & 30x^4_{n+\frac{1}{6}} & 42x^5_{n+\frac{1}{6}} & 56x^6_{n+\frac{1}{6}} \\ 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x^2_{n+\frac{1}{3}} & 20x^3_{n+\frac{1}{3}} & 30x^4_{n+\frac{1}{3}} & 42x^5_{n+\frac{1}{3}} & 56x^6_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} & 42x^5_{n+\frac{1}{2}} & 56x^6_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x^2_{n+\frac{2}{3}} & 20x^3_{n+\frac{2}{3}} & 30x^4_{n+\frac{2}{3}} & 42x^5_{n+\frac{2}{3}} & 56x^6_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6x_{n+\frac{5}{6}} & 12x^2_{n+\frac{5}{6}} & 20x^3_{n+\frac{5}{6}} & 30x^4_{n+\frac{5}{6}} & 42x^5_{n+\frac{5}{6}} & 56x^6_{n+\frac{5}{6}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}$$

$$= \begin{bmatrix} y_{n+\frac{2}{3}} \\ y_{n+\frac{5}{6}} \\ f_n \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{bmatrix} \qquad (3.3.56)$$

Solving (3.3.56) by Gaussian elimination method, the following parameters were obtained:

$$
\begin{aligned}
a_0 &= y_{n+\frac{2}{3}} - a_1 x_{n+\frac{2}{3}} - a_2 x^2_{n+\frac{2}{3}} - a_3 x^3_{n+\frac{2}{3}} - a_4 x^4_{n+\frac{2}{3}} - a_5 x^5_{n+\frac{2}{3}} - a_6 x^6_{n+\frac{2}{3}} - a_7 x^7_{n+\frac{2}{3}} - a_8 x^8_{n+\frac{2}{3}} \\
a_1 &= \frac{y_{n+\frac{5}{6}} - y_{n+\frac{2}{3}}}{\frac{1}{6}h} - a_2\left(x_{n+\frac{5}{6}} + x_{n+\frac{2}{3}}\right) - a_3\left(x^2_{n+\frac{5}{6}} + x_{n+\frac{5}{6}}x_{n+\frac{2}{3}} + x^2_{n+\frac{2}{3}}\right) \\
& -a_4\left(x^3_{n+\frac{5}{6}} + x^2_{n+\frac{5}{6}}x_{n+\frac{2}{3}} + x_{n+\frac{5}{6}}x^2_{n+\frac{2}{3}} + x^3_{n+\frac{2}{3}}\right) \\
& -a_5\left(x^4_{n+\frac{5}{6}} + x^3_{n+\frac{5}{6}}x_{n+\frac{2}{3}} + x^2_{n+\frac{5}{6}}x^2_{n+\frac{2}{3}} + x_{n+\frac{5}{6}}x^3_{n+\frac{2}{3}} + x^4_{n+\frac{2}{3}}\right) \\
& -a_6\left(x^5_{n+\frac{5}{6}} + x^4_{n+\frac{5}{6}}x_{n+\frac{2}{3}} + x^3_{n+\frac{5}{6}}x^2_{n+\frac{2}{3}} + x^2_{n+\frac{5}{6}}x^3_{n+\frac{2}{3}} + x_{n+\frac{5}{6}}x^4_{n+\frac{2}{3}} + x^5_{n+\frac{2}{3}}\right) \\
& -a_7\left(x^6_{n+\frac{5}{6}} + x^5_{n+\frac{5}{6}}x_{n+\frac{2}{3}} + x^4_{n+\frac{5}{6}}x^2_{n+\frac{2}{3}} + x^3_{n+\frac{5}{6}}x^3_{n+\frac{2}{3}} + x^2_{n+\frac{5}{6}}x^4_{n+\frac{2}{3}} + x_{n+\frac{5}{6}}x^5_{n+\frac{2}{3}} + x^6_{n+\frac{2}{3}}\right) \\
& -a_8\left(x^7_{n+\frac{5}{6}} + x^6_{n+\frac{5}{6}}x_{n+\frac{2}{3}} + x^5_{n+\frac{5}{6}}x^2_{n+\frac{2}{3}} + x^4_{n+\frac{5}{6}}x^3_{n+\frac{2}{3}} + x^3_{n+\frac{5}{6}}x^4_{n+\frac{2}{3}}\right. \\
& \left. +x^2_{n+\frac{5}{6}}x^5_{n+\frac{2}{3}} + x_{n+\frac{5}{6}}x^6_{n+\frac{2}{3}} + x^7_{n+\frac{2}{3}}\right) \\
a_2 &= \frac{1}{2}f_n - 3a_3x_n - 6a_4x^2_n - 10a_5x^3_n - 15a_6x^4_n + 21a_7x^5_n + 28a_8x^6_n \\
a_3 &= \frac{f_{n+\frac{1}{6}} - f_n}{h} - 2a_4\left(x_{n+\frac{1}{6}} + x_n\right) - \frac{10}{3}a_5\left(x^2_{n+\frac{1}{6}} + x_{n+\frac{1}{6}}x_n + x^2_n\right) \\
& -5a_6\left(x^3_{n+\frac{1}{6}} + x^2_{n+\frac{1}{6}}x_n + x_{n+\frac{1}{6}}x^2_n + x^3_n\right) - 7a_7\left(x^4_{n+\frac{1}{6}} + x^3_{n+\frac{1}{6}}x_n + x^2_{n+\frac{1}{6}}x^2_n + x_{n+\frac{1}{6}}x^3_n + x^4_n\right) \\
& -\frac{28}{3}a_8\left(x^5_{n+\frac{1}{6}} + x^4_{n+\frac{1}{6}}x_n + x^3_{n+\frac{1}{6}}x^2_n + x^2_{n+\frac{1}{6}}x^3_n + x_{n+\frac{1}{6}}x^4_n + x^5_n\right) \\
a_4 &= \frac{\frac{1}{6}f_{n+\frac{1}{3}} - \frac{1}{3}f_{n+\frac{1}{6}} + \frac{1}{6}f_n}{\frac{1}{9}h^2} - \frac{5}{3}a_5\left(x_{n+\frac{1}{3}} + x_{n+\frac{1}{6}} + x_n\right) \qquad (3.3.57) \\
& -\frac{5}{2}a_6\left(x^2_{n+\frac{1}{3}} + x_{n+\frac{1}{3}}x_{n+\frac{1}{6}} + x^2_{n+\frac{1}{6}} + x_{n+\frac{1}{3}}x_n + x_{n+\frac{1}{6}}x_n + x^2_n\right) \\
& -\frac{7}{2}a_7\left(x^3_{n+\frac{1}{3}} + x^2_{n+\frac{1}{3}}x_{n+\frac{1}{6}} + x_{n+\frac{1}{3}}x^2_{n+\frac{1}{6}} + x^3_{n+\frac{1}{6}} + x^2_{n+\frac{1}{3}}x_n + x_{n+\frac{1}{3}}x_{n+\frac{1}{6}}x_n\right. \\
& \left. +x^2_{n+\frac{1}{6}}x_n + x_{n+\frac{1}{3}}x^2_n + x_{n+\frac{1}{6}}x^2_n + x^3_n\right) \\
& -\frac{14}{3}a_8\left(x^4_{n+\frac{1}{3}} + x^3_{n+\frac{1}{3}}x_{n+\frac{1}{6}} + x^2_{n+\frac{1}{3}}x^2_{n+\frac{1}{6}} + x_{n+\frac{1}{3}}x^3_{n+\frac{1}{3}} + x^4_{n+\frac{1}{3}} + x^3_{n+\frac{1}{3}}x_n + x^2_{n+\frac{1}{3}}x_{n+\frac{1}{6}}x_n\right. \\
& \left. +x_{n+\frac{1}{3}}x^2_{n+\frac{1}{6}}x_n + x^3_{n+\frac{1}{6}}x_n + x^2_{n+\frac{1}{3}}x^2_n + x_{n+\frac{1}{3}}x_{n+\frac{1}{6}}x^2_n + x^2_{n+\frac{1}{6}}x^2_n + x_{n+\frac{1}{3}}x^3_n + x_{n+\frac{1}{6}}x^3_n + x^4_n\right) \\
a_5 &= \frac{\frac{1}{108}f_{n+\frac{1}{2}} - \frac{1}{36}f_{n+\frac{1}{3}} + \frac{1}{36}f_{n+\frac{1}{6}} - \frac{1}{108}f_n}{\frac{5}{972}h^3} - \frac{3}{2}a_6\left(x_{n+\frac{1}{2}} + x_{n+\frac{1}{3}} + x_{n+\frac{1}{6}} + x_n\right) \\
& -\frac{21}{10}a_7\left(x^2_{n+\frac{1}{2}} + x_{n+\frac{1}{2}}x_{n+\frac{1}{3}} + x^2_{n+\frac{1}{3}} + x_{n+\frac{1}{2}}x_{n+\frac{1}{6}} + x_{n+\frac{1}{3}}x_{n+\frac{1}{6}} + x^2_{n+\frac{1}{6}} + x_{n+\frac{1}{2}}x_n + x_{n+\frac{1}{3}}x_n\right. \\
& \left. +x_{n+\frac{1}{6}}x_n + x^2_n\right)
\end{aligned}
$$

$$-\frac{14}{5}a_8\Big(x^3_{n+\frac{1}{2}}+x^2_{n+\frac{1}{2}}x_{n+\frac{1}{3}}+x_{n+\frac{1}{2}}x^2_{n+\frac{1}{3}}+x^3_{n+\frac{1}{3}}+x^2_{n+\frac{1}{2}}x_{n+\frac{1}{6}}+x_{n+\frac{1}{2}}x_{n+\frac{1}{3}}x_{n+\frac{1}{6}}+x^2_{n+\frac{1}{3}}x_{n+\frac{1}{6}}$$
$$+x_{n+\frac{1}{2}}x^2_{n+\frac{1}{6}}+x_{n+\frac{1}{3}}x^2_{n+\frac{1}{6}}+x^3_{n+\frac{1}{6}}+x^2_{n+\frac{1}{2}}x_n+x_{n+\frac{1}{2}}x_{n+\frac{1}{6}}x_n+x^2_{n+\frac{1}{3}}x_n+x_{n+\frac{1}{2}}x_{n+\frac{1}{6}}x_n$$
$$+x_{n+\frac{1}{3}}x_{n+\frac{1}{6}}x_n+x^2_{n+\frac{1}{6}}x_n+x_{n+\frac{1}{2}}x^2_n+x_{n+\frac{1}{3}}x^2_n+x_{n+\frac{1}{6}}x^2_n+x^3_n\Big)$$

$$a_6 = \frac{\frac{1}{3888}f_{n+\frac{2}{3}}-\frac{1}{972}f_{n+\frac{1}{2}}+\frac{1}{648}f_{n+\frac{1}{3}}-\frac{1}{972}f_{n+\frac{1}{6}}+\frac{11}{3888}f_n}{\frac{5}{34992}h^4}$$
$$-\frac{7}{5}a_7\Big(x_{n+\frac{2}{3}}+x_{n+\frac{1}{2}}+x_{n+\frac{1}{3}}+x_{n+\frac{1}{6}}+x_n\Big)-\frac{28}{15}a_8\Big(x^2_{n+\frac{2}{3}}+x_{n+\frac{2}{3}}x_{n+\frac{1}{2}}+x^2_{n+\frac{1}{2}}+x_{n+\frac{2}{3}}x_{n+\frac{1}{3}}$$
$$+x_{n+\frac{1}{2}}x_{n+\frac{1}{3}}+x^2_{n+\frac{1}{3}}+x_{n+\frac{2}{3}}x_{n+\frac{1}{6}}+x_{n+\frac{1}{2}}x_{n+\frac{1}{6}}+x_{n+\frac{1}{3}}x_{n+\frac{1}{6}}+x^2_{n+\frac{1}{6}}+x_{n+\frac{2}{3}}x_n+x_{n+\frac{1}{2}}x_n$$
$$+x_{n+\frac{1}{3}}x_n+x_{n+\frac{1}{6}}x_n+x^2_n\Big)$$

$$a_7 = \frac{\frac{1}{209952}f_{n+\frac{5}{6}}-\frac{5}{209952}f_{n+\frac{2}{3}}+\frac{5}{104976}f_{n+\frac{1}{2}}-\frac{5}{104976}f_{n+\frac{1}{3}}+\frac{5}{209952}f_{n+\frac{1}{6}}-\frac{1}{209952}f_n}{\frac{35}{11337408}h^5}$$
$$-\frac{4}{3}a_8\Big(x_{n+\frac{5}{6}}+x_{n+\frac{2}{3}}+x_{n+\frac{1}{2}}+x_{n+\frac{1}{3}}+x_{n+\frac{1}{6}}+x_n\Big)$$
$$a_8 = \frac{\frac{5}{68024448}f_{n+1}-\frac{5}{11337408}f_{n+\frac{5}{6}}+\frac{25}{22674816}f_{n+\frac{2}{3}}-\frac{25}{17006112}f_{n+\frac{1}{2}}+\frac{25}{22674816}f_{n+\frac{1}{3}}-\frac{5}{11337408}f_{n+\frac{1}{6}}+\frac{5}{68024448}f_n}{\frac{175}{2754990144}h^6}$$

Substituting (3.3.57) into (3.3.53) with some manipulations leads to the continuous implicit hybrid one step method in the form of a CLMM described by the formula

$$y(x)=\sum_{j=0}^{k}\alpha_j(x)y_{n+j}+\sum_{\nu_i}\alpha_{\nu_i}(x)y_{n+\nu_i}+h^2\left[\sum_{j=0}^{k}\beta_j(x)f_{n+j}+\sum_{\nu_i}\beta_{\nu_i}(x)f_{n+\nu_i}\right],\ i=1,2,3,4,5 \quad (3.3.58)$$

Now, for $k=1, \nu_1=\frac{1}{6}, \nu_2=\frac{1}{3}, \nu_3=\frac{1}{2}, \nu_4=\frac{2}{3}$ and $\nu_5=\frac{5}{6}$, let $\alpha(x)$ and $\beta(x)$ be written as continuous functions of t then, we obtain the following coefficients of (3.3.58) as:

$$\alpha_{\frac{3}{5}}(t) = -6t$$

$$\alpha_{\frac{5}{6}}(t) = 6t+1$$

$$\beta_0(t) = \frac{h^2}{725760}(839808t^3 + 1679616t^7 + 1088640t^6 + 163296t^5 - 78624t^4 - 24192t^3 + 289t)$$

$$\beta_{\frac{1}{6}}(t) = \frac{-h^2}{13440}(93312t^8 + 207360t^7 + 145152t^6 + 24192t^5 - 10416t^4 - 3360t^3 + 41t)$$

$$\beta_{\frac{1}{3}}(t) = \frac{h^2}{80640}(1399680t^8 + 3421440t^7 + 2685312t^6 + 526176t^5 - 191520t^4 - 67200t^3 + 851t)$$

$$\beta_{\frac{1}{2}}(t) = -\frac{h^2}{181440}(4199040t^8 + 11197440t^7 + 10015488t^6 + 2612736t^5 - 710640t^4 -302400t^3 + 4157t) \qquad (3.3.59)$$

$$\beta_{\frac{2}{3}}(t) = \frac{h^2}{8960}(155520t^8 + 449280t^7 + 459648t^6 + 167328t^5 - 19040t^4 - 22400t^3 + 455t)$$

$$\beta_{\frac{5}{6}}(t) = \frac{-h^2}{40320}(279936t^8 + 870912t^7 + 1016064t^6 + 508032t^5 + 49392t^4 - 51744t^3 -20160t^2 - 1973t)$$

$$\beta_1(t) = \frac{h^2}{725760}(839808t^7 + 2799360t^7 + 3701376t^6 + 2449440t^5 + 828576t^4 + 120960t^3 - 995t)$$

Evaluating (3.3.58) at $x_n, x_{n+\frac{1}{6}}, x_{n+\frac{1}{3}}, x_{n+\frac{1}{2}}, x_{n+1}$ using (3.2.8) gives $t = -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, \frac{1}{6}$ and yields the following discrete methods from (3.3.59);

$$4y_{n+\frac{5}{6}} - 5y_{n+\frac{2}{3}} + y_n = \frac{h^2}{217728}\Big[-95f_{n+1} + 2334f_{n+\frac{5}{6}} + 21255f_{n+\frac{2}{3}} + 18532f_{n+\frac{1}{2}} + 11679f_{n+\frac{1}{3}} +6366_{n+\frac{1}{6}} + 409f_n\Big] \qquad (3.3.60a)$$

$$3y_{n+\frac{5}{6}} - 4y_{n+\frac{5}{6}} + y_{n+\frac{1}{6}} = -\frac{h^2}{362880}\Big[95f_{n+1} - 2754f_{n+\frac{5}{6}} - 26883f_{n+\frac{2}{3}} - 19708f_{n+\frac{1}{2}} - 10503f_{n+\frac{1}{3}} -738f_{n+\frac{1}{6}} + 11f_n\Big] \qquad (3.3.60b)$$

$$2y_{n+\frac{5}{6}} - 3y_{n+\frac{2}{3}} + y_{n+\frac{1}{3}} = \frac{h^2}{725760}\Big[-137f_{n+1} + 3762_{n+\frac{5}{6}} + 35073f_{n+\frac{2}{3}} + 19708f_{n+\frac{1}{2}} + 2313f_{n+\frac{1}{3}}$$

$$-270f_{n+\frac{1}{6}}+31f_n\Big] \qquad (3.3.60c)$$

$$y_{n+\frac{5}{6}}-2y_{n+\frac{2}{3}}+y_{n+\frac{1}{2}}=\frac{h^2}{2177280}\Big[-221f_{n+1}+5862f_{n+\frac{5}{6}}+49353f_{n+\frac{2}{3}}+5428f_{n+\frac{1}{2}}$$
$$+213f_{n+\frac{1}{3}}-186f_{n+\frac{1}{6}}+31f_n\Big] \qquad (3.3.60d)$$

$$y_{n+1}-2y_{n+\frac{5}{6}}+y_{n+\frac{2}{3}}=-\frac{h^2}{2177280}\Big[-4315f_{n+1}-53994f_{n+\frac{5}{6}}+2307f_{n+\frac{2}{3}}-7948f_{n+\frac{1}{2}}$$
$$+4827f_{n+\frac{1}{3}}-1578f_{n+\frac{1}{6}}+221f_n\Big] \qquad (3.3.60e)$$

Differentiating (3.3.59) yields the following:

$$\alpha'_{\frac{2}{3}}(t)=-\frac{6}{h}$$

$$\alpha'_{\frac{5}{6}}(t)=\frac{6}{h}$$

$$\beta'_0(t)=\frac{h}{725760}(6718464t^7+11757312t^6+6531840t^5+816480t^4-314496t^3-7257t^2+289)$$

$$\beta'_{\frac{1}{6}}(t)=-\frac{h}{13440}(746469t^7+1451520t^6+870912t^5+120960t^4-41664t^3-10080t^2+41)$$

$$\beta'_{\frac{1}{3}}(t)=\frac{h}{80640}(11197440t^7+23950080t^6+16111872t^5+2630880t^4-766080t^3$$
$$-201600t^2+851)$$

$$(3.3.61)$$

$$\beta'_{\frac{1}{2}}(t)=-\frac{h}{181440}(33592320t^7+78382080t^6+60092928t^5+13063680t^4-2842560t^3$$
$$-907200t^2+4157)$$

$$\beta'_{\frac{2}{3}}(t)=\frac{h}{40320}(2239488t^7+6096384t^6+6096384t^5+2540160t^4+197568t^3-155232t^2$$
$$-40320t-1973)$$

$$\beta'_1(t)=\frac{h}{725760}(6718464t^7+19595520t^6+22208256t^5+12247200t^4+3314304t^3$$
$$+362880t^2-995)$$

Evaluating (3.3.61) for $t = -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6}, 0$ and $\frac{1}{6}$ the following derivative methods were obtained:

$$
\begin{aligned}
hy'_n - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} &= -\frac{h^2}{241920}\Big[-585f_{n+1} + 6962f_{n+\frac{5}{6}} + 26465f_{n+\frac{2}{3}} + 58876f_{n+\frac{1}{2}} \\
&\quad +18697f_{n+\frac{1}{3}} + 58738f_{n+\frac{1}{6}} + 12287f_n\Big] \\
hy'_{n+\frac{1}{6}} - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} &= \frac{h^2}{725760}\Big[29f_{n+1} - 8262f_{n+\frac{5}{6}} - 119817f_{n+\frac{2}{3}} - 101620f_{n+\frac{1}{2}} \\
&\quad -149013f_{n+\frac{1}{3}} - 45990f_{n+\frac{1}{6}} + 1313f_n\Big] \\
hy'_{n+\frac{1}{3}} - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} &= \frac{-h^2}{725760}\Big[-571f_{n+1} + 12438f_{n+\frac{5}{6}} + 105219f_{n+\frac{2}{3}} + 134132f_{n+\frac{1}{2}} \\
&\quad +55035f_{n+\frac{1}{3}} - 4266f_{n+\frac{1}{6}} + 413f_n\Big] \\
hy'_{n+\frac{1}{2}} - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{6}} &= \frac{h^2}{241920}\Big[63f_{n+1} - 3074f_{n+\frac{5}{6}} - 39587f_{n+\frac{2}{3}} - 19708f_{n+\frac{1}{2}} \\
&\quad +2201f_{n+\frac{1}{3}} - 418f_{n+\frac{1}{6}} + 43f_n\Big] \qquad (3.3.62) \\
hy'_{n+\frac{2}{3}} - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} &= -\frac{h^2}{725760}\Big[-731f_{n+1} + 14742f_{n+\frac{5}{6}} + 57123f_{n+\frac{2}{3}} - 15884f_{n+\frac{1}{2}} \\
&\quad +6939f_{n+\frac{1}{3}} - 1962f_{n+\frac{1}{6}} + 253f_n\Big] \\
hy'_{n+\frac{5}{6}} - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} &= \frac{h^2}{725760}\Big[-995f_{n+1} + 35514f_{n+\frac{5}{6}} + 36855f_{n+\frac{2}{3}} - 16628f_{n+\frac{1}{2}} \\
&\quad +7659f_{n+\frac{1}{3}} - 2214f_{n+\frac{1}{6}} + 289f_n\Big] \\
hy'_{n+1} - 6y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} &= -\frac{h^2}{241920}\Big[-12393f_{n+1} - 55246f_{n+\frac{5}{6}} + 18689f_{n+\frac{2}{3}} - 19460f_{n+\frac{1}{2}} \\
&\quad +10921f_{n+\frac{1}{3}} - 3470f_{n+\frac{1}{6}} + 479f_n\Big]
\end{aligned}
$$

Combining (3.3.60) and (3.3.62) using (3.1.11) and (3.2.12) respectively gives

$$
\begin{bmatrix}
0 & 0 & 0 & -1088640 & 870912 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
362880 & 0 & 0 & -1451520 & 1088640 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 725760 & 0 & -2177280 & 1451520 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2177280 & -4354560 & 2177280 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2177280 & -4354560 & 2177280 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1451520 & -1451520 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4354560 & -4354560 & 0 & 725760h & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4354560 & -4354560 & 0 & 0 & 725760h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1451520 & -1451520 & 0 & 0 & 0 & 241920h & 0 & 0 & 0 \\
0 & 0 & 0 & 4354560 & -4354560 & 0 & 0 & 0 & 0 & 725760h & 0 & 0 \\
0 & 0 & 0 & 4354560 & -4354560 & 0 & 0 & 0 & 0 & 0 & 725760h & 0 \\
0 & 0 & 0 & 1451520 & -1451520 & 0 & 0 & 0 & 0 & 0 & 0 & 241920h
\end{bmatrix}
$$

$$\begin{bmatrix} y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{2}{3}} \\ y_{n+\frac{5}{6}} \\ y_{n+1} \\ y'_{n+\frac{1}{6}} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{2}{3}} \\ y'_{n+\frac{5}{6}} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} -217728 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -241920h \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \begin{bmatrix} 409 \\ -11 \\ 31 \\ 31 \\ -221 \\ -12287 \\ 1313 \\ -413 \\ 43 \\ -253 \\ 289 \\ -479 \end{bmatrix} [f_n]$$

$$
+\begin{bmatrix}
6366h^2 & 11679h^2 & 18532h^2 & 21255h^2 & 2334h^2 & -95h^2 \\
738h^2 & 10503h^2 & 19708h^2 & 26883h^2 & 2754h^2 & 95h^2 \\
-270h^2 & 2313h^2 & 19708h^2 & 35073h^2 & 3762h^2 & -137h^2 \\
-186h^2 & 213h^2 & 5428h^2 & 49353h^2 & 5862h^2 & -221h^2 \\
-1578h^2 & -4827h^2 & 7948h^2 & -2307h^2 & 53994h^2 & 4315h^2 \\
-58738h^2 & -18697h^2 & -58876h^2 & -26465h^2 & -6962h^2 & 585h^2 \\
-45990h^2 & -149013h^2 & -101620h^2 & -119817h^2 & -8262h^2 & 29h^2 \\
4266h^2 & -55035h^2 & -134132h^2 & -105219h^2 & -12438h^2 & 571h^2 \\
-418h^2 & 2201h^2 & -19708h^2 & -39587h^2 & -3074h^2 & 63h^2 \\
1962h^2 & -6939h^2 & 15884h^2 & -57123h^2 & -14742h^2 & 731h^2 \\
-2214h^2 & 7659h^2 & -16628h^2 & 36855h^2 & 35514h^2 & -995h^2 \\
3470h^2 & -10921h^2 & 19460h^2 & -18689h^2 & 55246h^2 & 12393h^2
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{6}} \\
f_{n+\frac{1}{3}} \\
f_{n+\frac{1}{2}} \\
f_{n+\frac{2}{3}} \\
f_{n+\frac{5}{6}} \\
f_{n+1}
\end{bmatrix}
\quad (3.3.63)
$$

and

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{2}{3}}\\ y_{n+\frac{5}{6}}\\ y_{n+1}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{3}}\\ y'_{n+\frac{1}{2}}\\ y'_{n+\frac{2}{3}}\\ y'_{n+\frac{5}{6}}\\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{6}h\\ 1 & \frac{1}{3}h\\ 1 & \frac{1}{2}h\\ 1 & \frac{2}{3}h\\ 1 & \frac{5}{6}h\\ 1 & h\\ & 1\\ & 1\\ & 1\\ & 1\\ & 1\\ & 1 \end{bmatrix} \begin{bmatrix} y_n\\ y'_n \end{bmatrix} + \begin{bmatrix} \frac{28549}{4354560}h^2\\ \frac{1027}{68040}h^2\\ \frac{253}{10752}h^2\\ \frac{272}{8505}h^2\\ \frac{35225}{870912}h^2\\ \frac{41}{840}h^2\\ \frac{19087}{362880}h\\ \frac{1139}{22680}h\\ \frac{137}{2688}h\\ \frac{143}{2835}h\\ \frac{3715}{7257}h\\ \frac{41}{840}h \end{bmatrix} [f_n]$$

$$
+\begin{bmatrix}
\frac{275}{20736}h^2 & -\frac{5717}{483840}h^2 & \frac{10621}{1088640}h^2 & -\frac{7703}{1451520}h^2 & \frac{403}{241920}h^2 & \frac{-199}{870912}h^2 \\
\frac{97}{1890}h^2 & -\frac{2}{81}h^2 & \frac{197}{8505}h^2 & \frac{97}{7560}h^2 & \frac{23}{5670}h^2 & -\frac{19}{34120}h^2 \\
\frac{165}{1792}h^2 & -\frac{267}{17920}h^2 & \frac{5}{128}h^2 & -\frac{363}{17920}h^2 & \frac{57}{8960}h^2 & -\frac{47}{53760}h^2 \\
\frac{376}{2835}h^2 & -\frac{2}{945}h^2 & \frac{656}{8505}h^2 & -\frac{2}{81}h^2 & \frac{8}{945}h^2 & -\frac{2}{1701}h^2 \\
\frac{8375}{48384}h^2 & \frac{3125}{290304}h^2 & \frac{25625}{217728}h^2 & -\frac{625}{96768}h^2 & \frac{275}{20736}h^2 & -\frac{1375}{870912}h^2 \\
\frac{3}{14}h^2 & \frac{3}{140}h^2 & \frac{17}{105}h^2 & \frac{3}{280}h^2 & \frac{3}{70}h^2 & 0 \\
\frac{2703}{15120}h & -\frac{15487}{120960}h & \frac{293}{2835}h & -\frac{6737}{120960}h & \frac{263}{15120}h & -\frac{863}{362880}h \\
\frac{47}{189}h & \frac{11}{7560}h & \frac{166}{2835}h & -\frac{269}{7560}h & \frac{11}{945}h & -\frac{37}{22680}h \\
\frac{27}{112}h & \frac{387}{4480}h & \frac{17}{105}h & -\frac{243}{4480}h & \frac{9}{560}h & -\frac{29}{13440}h \\
\frac{232}{945}h & \frac{64}{945}h & \frac{752}{2835}h & \frac{29}{945}h & \frac{8}{945}h & -\frac{4}{2835}h \\
\frac{725}{3024}h & \frac{2125}{24192}h & \frac{125}{567}h & \frac{3875}{24192}h & \frac{235}{3024}h & -\frac{275}{72576}h \\
\frac{9}{25}h & \frac{9}{280}h & \frac{34}{105}h & \frac{9}{280}h & \frac{9}{35}h & \frac{41}{840}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{5}{6}} \\ f_{n+1}
\end{bmatrix}
\quad (3.3.64)
$$

Equation (3.3.64) can be written explicitly as

$$\begin{aligned}
y_{n+\frac{1}{6}} &= y_n + \frac{1}{6}hy'_n + \frac{h^2}{4354560}\Big[-995f_{n+1} + 7254f_{n+\frac{5}{6}} - 23109f_{n+\frac{2}{3}} + 42484f_{n+\frac{1}{2}} \\
&\quad -51453f_{n+\frac{1}{3}} + 57750f_{n+\frac{1}{6}} + 28549f_n\Big] \\
y_{n+\frac{1}{3}} &= y_n + \frac{1}{3}hy'_n + \frac{h^2}{68040}\Big[-38f_{n+1} + 276f_{n+\frac{5}{6}} - 873f_{n+\frac{2}{3}} + 1576f_{n+\frac{1}{2}} \\
&\quad -1680f_{n+\frac{1}{3}} + 3492f_{n+\frac{1}{6}} + 1027f_n\Big] \\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + \frac{h^2}{53760}\Big[-47f_{n+1} + 342f_{n+\frac{5}{6}} - 1089f_{n+\frac{2}{3}} + 2100f_{n+\frac{1}{2}} \\
&\quad -801f_{n+\frac{1}{3}} + 4950f_{n+\frac{1}{6}} + 1265f_n\Big] \\
y_{n+\frac{2}{3}} &= y_n + \frac{2}{3}hy'_n + \frac{h^2}{8505}\Big[-10f_{n+1} + 72f_{n+\frac{5}{6}} - 210f_{n+\frac{2}{3}} + 656f_{n+\frac{1}{2}} - 18f_{n+\frac{1}{3}} \\
&\quad +1128f_{n+\frac{1}{6}} + 272f_n\Big] \\
y_{n+\frac{5}{6}} &= y_n + \frac{5}{6}hy'_n + \frac{h^2}{870912}\Big[-1375f_{n+1} + 11550f_{n+\frac{5}{6}} - 5625f_{n+\frac{2}{3}} + 102500f_{n+\frac{1}{2}} \\
&\quad +9375f_{n+\frac{1}{3}} + 150750f_{n+\frac{1}{6}} + 35225f_n\Big] \\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{840}\Big[36f_{n+\frac{5}{6}} + 9f_{n+\frac{2}{3}} + 136f_{n+\frac{1}{2}} + 18f_{n+\frac{1}{2}} + 180f_{n+\frac{1}{6}} + 41f_n\Big] \qquad (3.3.65) \\
y'_{n+\frac{1}{6}} &= y'_n + \frac{h}{362880}\Big[-863f_{n+1} + 6312f_{n+\frac{5}{6}} - 20211f_{n+\frac{2}{3}} + 37504f_{n+\frac{1}{2}} \\
&\quad -46461f_{n+\frac{1}{3}} + 65112f_{n+\frac{1}{6}} + 19087f_n\Big] \\
y'_{n+\frac{1}{3}} &= y'_n + \frac{h}{22680}\Big[-37f_{n+1} + 264f_{n+\frac{5}{6}} - 807f_{n+\frac{2}{3}} + 1328f_{n+\frac{1}{2}} + 33f_{n+\frac{1}{3}} \\
&\quad +5640f_{n+\frac{1}{6}} + 1139f_n\Big] \\
y'_{n+\frac{1}{2}} &= y'_n + \frac{h}{13440}\Big[-29f_{n+1} + 216f_{n+\frac{5}{6}} - 729f_{n+\frac{2}{3}} + 2176f_{n+\frac{1}{2}} + 1161f_{n+\frac{1}{3}} \\
&\quad +3240f_{n+\frac{1}{6}} + 685f_n\Big] \\
y'_{n+\frac{2}{3}} &= y'_n + \frac{h}{2835}\Big[-4f_{n+1} + 24f_{n+\frac{5}{6}} + 87f_{n+\frac{2}{3}} + 752f_{n+\frac{1}{2}} \\
&\quad +192f_{n+\frac{1}{3}} + 696f_{n+\frac{1}{6}} + 143f_n\Big] \\
y'_{n+\frac{5}{6}} &= y'_n + \frac{h}{72576}\Big[-275f_{n+1} + 5640f_{n+\frac{5}{6}} + 11625f_{n+\frac{2}{3}} + 16000f_{n+\frac{1}{2}} \\
&\quad +6375f_{n+\frac{1}{3}} + 17400f_{n+\frac{1}{6}} + 3715f_n\Big] \\
y'_{n+1} &= y'_n + \frac{h}{840}\Big[41f_{n+1} + 216f_{n+\frac{5}{6}} + 27f_{n+\frac{2}{3}} + 272f_{n+\frac{1}{2}} + 27f_{n+\frac{1}{3}} \\
&\quad +216f_{n+\frac{1}{6}} + 41f_n\Big]
\end{aligned}$$

Chapter 4

Analysis of the Methods

4.1 Introduction

Basic properties of the main methods and their associated block method, are analysed to establish their validity. These properties, namely: order, error constant, consistency and zero stability reveal the nature of convergence of the methods. The regions of absolute stability of the methods have also been obtained in this chapter. In what follows, a brief introduction of these properties is made for a better comprehension of the chapter.

4.1.1 Order and Error Constant

4.1.1.1 Order of the method

Let the linear difference operator $\mathcal{L}$ associated with the continuous implicit one step hybrid method (3.2.7) be defined as

$$\begin{aligned}\mathcal{L}[y(x);h] &= \sum_{j=0}^{k}\left\{\alpha_j y(x_n+jh) - \alpha_{\nu_i} y(x_n+\nu_i h) - h^2\beta_j y''(x_n+jh)\right.\\ &\quad \left.-h^2\beta_{\nu_i} y''({}_x n+jh)\right\};\quad i=1,2,\ldots,m \qquad (4.1.1)\end{aligned}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. Expanding $y(x_n+jh)$ and $y''(x_n+jh)$, $j=0,\nu_i,1;\ \ i=1,2,\ldots,m$ in Taylor series about x_n and collecting like terms in h and y gives;

$$\mathcal{L}[y(x);h] = C_0 y(x) + C_1 h y^{(1)}(x) + C_2 h^2 y^{(2)}(x) + \cdots + C_p h^p y^{(p)}(x) + \cdots \qquad (4.1.2)$$

Definition 4.1.1

The difference operator $\mathcal{L}$ and the associated continuous implicit hybrid one step method (3.2.7) are said to be of order p if in (4.1.2) $C_0 = C_1 = C_2 = \cdots = C_p = C_{p+1} = 0, C_{p+2} \neq 0$.

Definition 4.1.2

The term C_{p+2} is called the error constant and it implies that the local truncation error is given by

$$t_{n+k} = C_{p+2}h^{p+2}y^{(p+2)}(x_n) + O(h^{p+3})$$

4.1.1.2 Order of the Block

The order of the block will be defined following the method of Chollom *et.al.* (2007) however, with some modification to accommodate general higher order ordinary differential equations and offstep points.

Let the implicit block hybrid one step method be defined by

$$\sum_j \alpha_{ij}^{(\mu)}(t)y_{n+j} = h^2 \sum_j \beta_{ij}^{(\mu)}(t)f_{n+j}, \quad i,j = 0, \nu_1, \ldots, \nu_m, k \tag{4.1.3}$$

where μ is the degree of the derivative of the continuous coefficients $\alpha_{ij}(t)$ and $\beta_{ij}(t)$ and m is the number of offstep points used.

In matrix form,(4.1.3) is equivalent to

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0\nu_m} & \alpha_{0k} \\ \alpha_{\nu_1 0} & \alpha_{\nu_1 1} & \cdots & \alpha_{\nu_1\nu_m} & \alpha_{\nu_1 k} \\ \vdots & \vdots & \vdots & \vdots & \\ \alpha_{\nu_m 0} & \alpha_{\nu_m 1} & \cdots & \alpha_{\nu_m\nu_m} & \alpha_{\nu_m k} \\ \alpha_{k0} & \alpha_{k1} & \cdots & \alpha_{k\nu_m} & \alpha_{kk} \\ \alpha'_{00} & \alpha'_{01} & \cdots & \alpha'_{0\nu_m} & \alpha'_{0k} \\ \alpha'_{\nu_1 0} & \alpha'_{\nu_1 1} & \cdots & \alpha'_{\nu_1\nu_m} & \alpha'_{\nu_1 k} \\ \vdots & \vdots & \vdots & \vdots & \\ \alpha'_{\nu_m 0} & \alpha'_{\nu_m 1} & \cdots & \alpha'_{\nu_m\nu_m} & \alpha'_{\nu_m k} \\ \alpha'_{k0} & \alpha'_{k1} & \cdots & \alpha'_{k\nu_m} & \alpha'_{kk} \end{bmatrix} \begin{bmatrix} y_n \\ y_{n+\nu_1} \\ \vdots \\ y_{n+\nu_m} \\ y_{n+1} \\ y'_n \\ y'_{n+\nu_1} \\ \vdots \\ y'_{n+\nu_m} \\ y'_{n+k} \end{bmatrix} = \begin{bmatrix} \beta_{00} & \beta_{01} & \cdots & \beta_{0\nu_m} & \beta_{0k} \\ \beta_{\nu_1 0} & \beta_{\nu_1 1} & \cdots & \beta_{\nu_1\nu_m} & \beta_{\nu_1 k} \\ \vdots & \vdots & \vdots & \vdots & \\ \beta_{\nu_m 0} & \beta_{\nu_m 1} & \cdots & \beta_{\nu_m\nu_m} & \beta_{\nu_m k} \\ \beta_{k0} & \beta_{k1} & \cdots & \beta_{k\nu_m} & \beta_{kk} \\ \beta'_{00} & \beta'_{01} & \cdots & \beta'_{0\nu_m} & \beta'_{0k} \\ \beta'_{\nu_1 0} & \beta'_{\nu_1 1} & \cdots & \beta'_{\nu_1\nu_m} & \beta'_{\nu_1 k} \\ \vdots & \vdots & \vdots & \vdots & \\ \beta'_{\nu_m 0} & \beta'_{\nu_m 1} & \cdots & \beta'_{\nu_m\nu_m} & \beta'_{\nu_m k} \\ \beta'_{k0} & \beta'_{k1} & \cdots & \beta'_{k\nu_m} & \beta'_{kk} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\nu_1} \\ \vdots \\ f_{n+\nu_m} \\ f_{n+k} \end{bmatrix}.$$

Let

$$\begin{aligned} \bar{\alpha}_0 &= [\alpha_{00}, \alpha_{\nu_1 0}, \ldots, \alpha_{\nu_m 0}, \alpha_{k0}, \alpha'_{00}, \alpha'_{\nu_1 0}, \ldots, \alpha'_{\nu_m 0}, \alpha'_{k0}]^T \\ \bar{\alpha}_{\nu_1} &= [\alpha_{01}, \alpha_{\nu_1 1}, \ldots, \alpha_{\nu_m 1}, \alpha_{k1}, \alpha'_{01}, \alpha'_{\nu_1 1}, \ldots, \alpha'_{\nu_m 1}, \alpha'_{k1}]^T \\ &\vdots \\ \bar{\alpha}_{\nu_m} &= [\alpha_{01}, \alpha_{\nu_1 1}, \ldots, \alpha_{\nu_m 1}, \alpha_{k1}, \alpha'_{01}, \alpha'_{\nu_1 1}, \ldots, \alpha'_{\nu_m 1}, \alpha'_{k1}]^T \\ \bar{\alpha}_k &= [\alpha_{0k}, \alpha_{1k}, \ldots, \alpha_{\nu_m k}, \alpha_{kk}, \alpha'_{0k}, \alpha'_{\nu_1 k}, \ldots, \alpha'_{\nu_m k}, \alpha'_{kk}]^T. \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{\beta}_0 &= [\beta_{00}, \beta_{\nu_1 0}, \ldots, \beta_{\nu_m 0}, \beta_{k0}, \beta'_{00}, \beta'_{\nu_1 0}, \ldots, \beta'_{\nu_m 0}, \beta'_{k0}]^T \\ \bar{\beta}_1 &= [\beta_{01}, \beta_{\nu_1 1}, \ldots, \beta_{\nu_m 1}, \beta_{k1}, \beta'_{01}, \beta'_{\nu_1 1}, \ldots, \beta'_{\nu_m 1}, \beta'_{k1}]^T \\ &\vdots \\ \bar{\beta}_{\nu_m} &= [\beta_{01}, \beta_{\nu_1 1}, \ldots, \beta_{\nu_m 1}, \beta_{k1}, \beta'_{01}, \beta'_{\nu_1 1}, \ldots, \beta'_{\nu_m 1}, \beta'_{k1}]^T \\ \bar{\beta}_k &= [\beta_{0k}, \beta_{1k}, \ldots, \beta_{\nu_m k}, \beta_{kk}, \beta'_{0k}, \beta'_{\nu_1 k}, \ldots, \beta'_{\nu_m k}, \beta'_{kk}]^T. \end{aligned}$$

Then, the linear difference operator, $\mathcal{L}$, associated with the implicit block hybrid one step method (4.1.3) is described by the formula

$$\mathcal{L}[y(x);h] = \sum_j \left[\bar{\alpha}_j y(x_n + jh) - h^2\bar{\beta}_j y''(x_n + jh)\right], \quad j = 0, \nu_1, \ldots, \nu_m, k, \tag{4.1.4}$$

where $y(x)$ is an arbitrary test function continuously differentiable on $[a, b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh)$, $j = 0, \nu_1, \ldots, \nu_m, k$ in Taylor series and collecting terms in (4.1.4) gives

$$\mathcal{L}[y(x);h] = \bar{c}_0 y(x) + \bar{c}_1 h y^{(1)}(x) + \bar{c}_2 h^2 y^{(2)}(x) + \cdots + \bar{c}_p h^p y^{(p)}(x) \tag{4.1.5}$$

where the $\bar{c}_i,\ i = 0, 1, \ldots, p$ are vectors.

Definition 4.1.3

The one-step implicit hybrid block linear method (4.1.3) and the associated linear difference operator (4.1.4) are said to have order q if $\bar{c}_0 = \bar{c}_1 = \cdots = \bar{c}_p = \bar{c}_{p+1} = 0$ and $\bar{c}_{p+2} \neq 0$.

Definition 4.1.4

The term $\bar{c}_{p+2}$ is called the error constant and implies that the local truncation error for the implicit block hybrid formula is given by

$$t_{n+k} = \bar{c}_{p+2} h^{p+2} y^{(p+2)}(x_n) + O(h^{p+3}) \tag{4.1.6}$$

4.1.2 Consistency

Definition 4.1.5

Given a continuous implicit one step hybrid method (3.2.7), the first and second characteristic polynomials are defined as

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \tag{4.1.7}$$

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j \tag{4.1.8}$$

where z is the principal root, $\alpha_k \neq 0$ and $\alpha_0^2 + \beta_0^2 \neq 0$.

Definition 4.1.6

The continuous implicit one step hybrid method (3.2.7) is said to be consistent if it satisfies the following conditions:

(i) the order $p \geq 1$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\rho(1) = \rho'(1) = 0$ and

(iv) $\rho''(1) = 2!\sigma(1)$

Remark

Condition (i) is a sufficient condition for the associated block method to be consistent i.e., $p \geq 1$ (Jator, 2007).

4.1.3 Zero Stability

Definition 4.1.7

The continuous implicit one step hybrid method (3.2.7) is said to be zero-stable if no root of the first characteristic polynomial $\rho(z)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than one.

Definition 4.1.8

The implicit hybrid block method (3.2.12) is said to be zero stable if the roots $z_s, s = 1, \ldots, n$ of the first characteristic polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z) = \det[z\bar{A} - \bar{E}] \tag{4.1.9}$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$.

4.1.4 Convergence

The convergence of the continuous implicit hybrid one step method (3.2.7) is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahlquist (Henrici,1962) for linear multistep methods. In what follows, we state Dahlquist's theorem without proof.

Theorem 4.1.1

The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

Remark

The numerical methods derived here are considered to be convergent in the limit as $h \to 0$ by Theorem 4.1.1.

4.1.5 Region of Absolute Stability

4.1.5.1 Region of absolute stability of the main methods

Consider the stability polynomial

$$\Pi(z, \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0 \tag{4.1.10}$$

where $\bar{h} = h^2\lambda^2$ and $\lambda = \dfrac{df}{dy}$ are assumed constant.
The polynomial equation (4.1.10) is obtained by applying the continuous implicit hybrid one step method (3.2.7) to the scalar test problem;

$$y'' = -\lambda^2 y \tag{4.1.11}$$

Definition 4.1.9

The method (3.2.7) is said to be absolutely stable if for a given $\bar{h}$ all the roots z_s of (4.1.10) satisfy $|z_s| < 1$, $s = 1, 2, \ldots, n$, where $\bar{h} = \lambda h$.

Remark

The interval of absolute stability is determined by the coefficient of the method (3.2.7).

Definition 4.1.10

The set $\Omega = \bar{h}(\theta)$ of points in the $\bar{h}$-plane for which the global error

$$e_{n+k} = |y_{n+k} - y(x_{n+k})| \tag{4.1.12}$$

remains bounded is called the interval of absolutely stability.

Remark

Since the roots of the stability polynomial (4.1.10) are complex numbers, we regard $\bar{h}$ as a complex number and define also a region of absolute stability.

To determine the region of absolute stability in this work, a method that requires neither the computation of roots of a polynomial nor the solving of simultaneous inequalities was adopted. This method according to Lambert (1973) is called the Boundary Locus Method (BLM).

Definition 4.1.11

The region $\mathcal{R}$ of the complex $\bar{h}$-plane such that the roots of $\Pi(r, \bar{h}) = 0$ lie within the unit circle whenever $\bar{h}$ lies in the interior of the region is called the region of absolute stability.

Remark

Let $\delta\mathcal{R}$ be the boundary of the region $\mathcal{R}$. Since the roots of the stability polynomial are continuous functions of $\bar{h}$, $\bar{h}$ will lie on $\delta\mathcal{R}$ when one of the roots of $\Pi(z, \bar{h}) = 0$ lies on the boundary of the unit circle. Thus, we redefine (4.1.10) in terms of Euler's number, $\exp i\theta$, as follows

$$\pi(\exp(i\theta), h) = \rho(\exp(i\theta)) - \bar{h}\sigma(\exp(i\theta)) = 0 \tag{4.1.13}$$

So that, the locus of the boundary $\delta\mathcal{R}$ is given by

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}. \tag{4.1.14}$$

In the next section, the basic properties of the methods are discussed.

4.2 Analysis of the One Step Method with One Offstep Point

In this section, the order, error constant, consistency and zero stability of the main method, equation (3.3.8) and the associated block method (3.3.12) are obtained. The region of absolute stability of the method (3.3.8) is also obtained.

4.2.1 Order and error constant

4.2.1.1 Order and error constant of the main method (3.3.8)

Writing (3.3.8) in the form

$$y_{n+1} - 2y_{n+\frac{1}{2}} + y_n - h^2\left[\frac{1}{48}f_n + \frac{5}{24}f_{n+\frac{1}{2}} + \frac{1}{48}f_{n+1}\right] = 0 \tag{4.2.1}$$

and expanding $y(x_n + jh)$ and $y''(x_n + jh),\ j = 1, \frac{1}{2}$ yields

$$\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j)} - 2\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j\frac{h^j}{j!}y_n^{(j)} + y_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{5}{24}\left(\frac{1}{2}\right)^j + \frac{1}{48}\right] - \frac{h^2}{48}y_n^{(2)} = 0 \tag{4.2.2}$$

Collecting terms in powers of h and y gives the following constants

$$
\begin{aligned}
c_0 &= 1-2+1=0\\
c_1 &= 1-1=0\\
c_2 &= \frac{1}{2}-\frac{1}{4}-\frac{10}{48}-\frac{1}{48}-\frac{1}{48}=0\\
c_3 &= \frac{1}{6}-\frac{1}{24}-\frac{1}{48}-\frac{5}{48}=0\\
c_4 &= \frac{1}{24}-\frac{1}{192}-\frac{1}{96}-\frac{5}{192}=0\\
c_5 &= \frac{1}{120}-\frac{1}{1920}-\frac{1}{288}-\frac{5}{1152}=0\\
c_6 &= \frac{1}{720}-\frac{23040}{-}\frac{1}{1152}-\frac{1}{221184}=\frac{401609}{849231360}=-4.7291\times10^{-4}
\end{aligned}
$$

Hence, the main method is of order $p=4$ with error constant $c_{p+2}=-4.7291\times10^{-4}$.

4.2.1.2 Order of the block method (3.3.12)

Let (3.3.12) be expressed in the form

$$
\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}
\begin{bmatrix}y_{n+\frac{1}{2}}\\ y_{n+1}\\ y'_{n+\frac{1}{2}}\\ y'_{n+1}\end{bmatrix}
-\begin{bmatrix}1&\frac{h}{2}\\1&h\\0&1\\0&1\end{bmatrix}
\begin{bmatrix}y_n\\ y'_n\end{bmatrix}
-\begin{bmatrix}\frac{7h^2}{96}\\ \frac{h^2}{6}\\ \frac{5h}{24}\\ \frac{h}{6}\end{bmatrix}
[f_n]-
\begin{bmatrix}\frac{h^2}{16}&-\frac{h^2}{96}\\ \frac{h^2}{3}&0\\ \frac{h}{3}&-\frac{h}{24}\\ \frac{2h}{3}&\frac{h}{6}\end{bmatrix}
\begin{bmatrix}f_{n+\frac{1}{2}}\\ f_{n+1}\end{bmatrix}=0 \quad (4.2.3)
$$

Expand (4.2.3) in Taylor series about x_n in the form

$$
\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(\frac{1}{2})^j h^j}{j!}y_n^{(j)}-y_n-\frac{h}{2}y_n^{(1)}-\frac{7h^2}{96}y_n^{(2)}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{1}{16}\left(\frac{1}{2}\right)^j-\frac{1}{96}\right]\\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j)}-y_n-hy_n^{(1)}-\frac{h^2}{6}y_n^{(2)}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{1}{3}\left(\frac{1}{2}\right)^j-0\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}h\right)^j}{j!}y_n^{(j+1)}-y_n^{(1)}-\frac{5h}{24}y_n^{(2)}-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+2)}\left[\frac{1}{3}\left(\frac{1}{2}\right)^j-\frac{1}{24}\right]\\
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j+1)}-y_n^{(1)}-\frac{h}{6}y_n^{(2)}-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{(j+2)}\left[\frac{2}{3}\left(\frac{1}{2}\right)^j-\frac{1}{6}\right]
\end{bmatrix}
=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad (4.2.4)
$$

$$\bar{c}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{c}_1 = \begin{bmatrix} \frac{1}{2}-\frac{1}{2} \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_2 = \begin{bmatrix} \left(\frac{1}{2}\right)^2 \frac{1}{2!} - \frac{7}{96} - \left[\frac{1}{16} - \frac{1}{96}\right] \\ \frac{1}{2!} - \frac{1}{6} - \left[\frac{1}{3} - 0\right] \\ \left(\frac{1}{2}\right) - \frac{5}{24} - \left[\frac{1}{3} - \frac{1}{24}\right] \\ 1 - \frac{1}{6} - \left[\frac{2}{3} + \frac{1}{6}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_3 = \begin{bmatrix} \left(\frac{1}{2}\right)^3 \frac{1}{3!} - \left[\frac{1}{16}\left(\frac{1}{2}\right) - \frac{1}{96}\right] \\ \frac{1}{3!} - \left[\frac{1}{3}\left(\frac{1}{2}\right) - 0\right] \\ \left(\frac{1}{2}\right)^2 \frac{1}{2!} - \left[\frac{1}{3}\left(\frac{1}{2}\right) - \frac{1}{24}\right] \\ \frac{1}{2!} - \left[\frac{2}{3}\left(\frac{1}{2}\right) + \frac{1}{6}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_4 = \begin{bmatrix} \left(\frac{1}{2}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{1}{16}\left(\frac{1}{2}\right)^2 - \frac{1}{96}\right] \\ \frac{1}{4!} - \frac{1}{2!}\left[\frac{1}{3}\left(\frac{1}{2}\right)^2 - 0\right] \\ \left(\frac{1}{2}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{1}{3}\left(\frac{1}{2}\right)^2 - \frac{1}{24}\right] \\ \frac{1}{3!} - \frac{1}{2!}\left[\frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_5 = \begin{bmatrix} \left(\frac{1}{2}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{1}{16}\left(\frac{1}{2}\right)^3 - \frac{1}{96}\right] \\ \frac{1}{5!} - \frac{1}{3!}\left[\frac{1}{3}\left(\frac{1}{2}\right)^3 - 0\right] \\ \left(\frac{1}{2}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{24}\right] \\ \frac{1}{4!} - \frac{1}{3!}\left[\frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{6}\right] \end{bmatrix} = \begin{bmatrix} \frac{8}{11520} \\ \frac{1}{720} \\ \frac{1}{384} \\ 0 \end{bmatrix}$$

Hence, the block method has order $p = (3,3,3,3)^T$ with error constant $c_{p+2} =$

$(\frac{8}{11520}, \frac{1}{720}, \frac{1}{384}, 0)^T$.

4.2.2 Consistency

4.2.2.1 Consistency of the main method (3.3.8)

The first and second characteristic polynomials of method (3.3.8) are given by

$$\rho(z) = z - 2z^{1/2} + 1$$

and

$$\sigma(z) = \frac{z + 10r^{1/2} + 1}{48}$$

By definition (4.1.6), the method (3.3.8) is consistent since it satisfies the following:

(i) the order of the method is $p = 4 \geq 1$.

(ii) $\alpha_0 = 1, \alpha_{\frac{1}{2}} = -2$ and $\alpha_1 = 1$. Thus, $\sum_j \alpha_j = 1 - 2 + 1 = 0, \;\; j = 0, \frac{1}{2}, 1$

(iii) $\rho(1) = 1 - 2 - 1 = 0$

$\rho'(z) = 1 - r^{-\frac{1}{2}}$

For $r = 1$

$\rho'(1) = 1 - 1 = 0$

$\therefore \;\; \rho(1) = \rho'(1) = 0$

(iv) $\rho''(z) = \frac{1}{2}z^{-3/2}$

$\rho''(1) = \frac{1}{2}$

$\sigma(1) = \frac{1+10+1}{48} = \frac{1}{4}$

$2!\sigma(1) = 2\left(\frac{1}{4}\right) = \frac{1}{2}$

$\therefore \;\; \rho''(1) = 2!\sigma(1)$

Similarly, the block method (3.3.12) is consistent by condition (i) of definition (4.1.6).

4.2.3 Zero Stability of One Step Method with One Offstep Point

4.2.3.1 Zero stability of the block method (3.3.12)

From (3.3.12) using the definitions in (3.2.12) as $h \to 0$

$$\begin{aligned} \rho(z) &= \det \left[z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \\ &= \det \begin{bmatrix} z & -1 & 0 & 0 \\ 0 & z-1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z-1 \end{bmatrix} = z^3(z-1) \end{aligned}$$

thus solving for z in

$$z^3(z-1) = 0 \qquad (4.2.5)$$

gives $z = 0$ or $z = 1$. Hence the block method is stable.

4.2.3.2 Zero stability of main method (3.3.8)

The first characteristic polynomial of (3.3.8) is given by

$$\rho(z) = z - 2z^{1/2} + 1 \qquad (4.2.6)$$

equating (4.2.6) to zero and solving for z gives

$$(\sqrt{z} - 1)^2 = 0$$

$$\Rightarrow \; z = 1$$

The root z of (4.2.6) for which $|z| = 1$ is simple, hence the method is zero stable as $h \to 0$ in the limit by definition (4.1.7) and by the stability of the block method (3.3.12).

4.2.4 Convergence

Following Theorem 4.1.1, the method (3.3.8) is convergent since it satisfies the necessary and sufficient conditions of consistency and zero stability.

4.2.5 Region of Absolute Stability of the one step Method with One Offstep Point

From (3.3.8), the first and second characteristic polynomial are as follows

$$\rho(z) = z - 2z^{1/2} + 1. \tag{4.2.7}$$

$$\sigma(z) = \frac{1}{48}z + \frac{10}{48}z^{1/2} + \frac{1}{48} \tag{4.2.8}$$

so that the boundary of the region of absolute stability is

$$\bar{h}(z) = \frac{\rho(z)}{\sigma(z)} = \frac{48(z - 2z^{1/2} + 1)}{z + 10z^{1/2} + 1} \tag{4.2.9}$$

Let $z = e^{i\theta}$, therefore (4.2.9) becomes

$$\bar{h}(\theta) = \frac{48(e^{i\theta} - 2e^{i\frac{1}{2}\theta} + 1)}{e^{i\theta} + 10e^{i\frac{1}{2}\theta} + 1} \tag{4.2.10}$$

for $e^{i\theta} = \cos\theta + i\sin\theta$, (4.2.10)reduces after some manipulation to

$$\bar{h}(\theta) = \frac{768\cos\frac{1}{2}\theta + 96\cos\theta - 864}{40\cos\frac{1}{2}\theta + 2\cos\theta + 102} \tag{4.2.11}$$

Evaluating (4.2.11) at intervals of 30^0 gives the following results;

Table 1. The boundaries of the region of absolute stability of the one step 1 offstep point method.

θ	0	30^0	60^0	90^0	120^0	150^0	180^0
$\bar{h}(\theta)$	0	-0.27	-1.10	-2.46	-4.36	-6.77	-9.60

Thus, the interval of absolute stability from table 1 is (-9.6,0). The region of absolute stability is shown in figure 4.1

Remark

The locus is symmetric about the x-axis that is, $x(-\theta) = x(\theta)$.

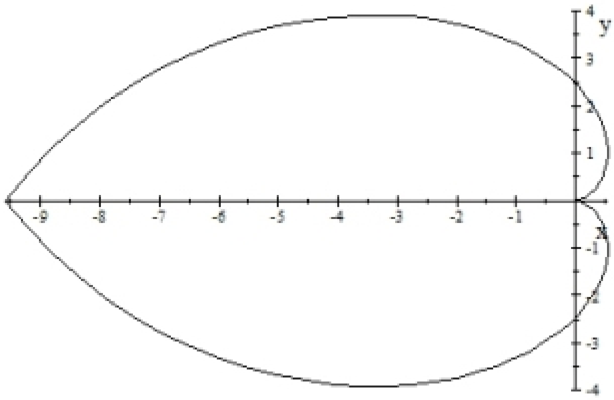

Figure 4.1: Region of Absolute stability of the One-Step Method with One offstep point

4.3 Analysis of the One Step Method with Two Offstep Points

In this section, the order, error constant, consistency and zero stability of the method (3.3.21a) and its associated block method (3.3.25) are obtained. The region of absolute stability of the method (3.3.21a) is also obtained.

4.3.1 Order and Error Constant

4.3.1.1 Order and error constant of the main method (3.3.21a)

Let (3.3.21a) be written in the form

$$y_{n+1} - 2y_{n+\frac{2}{3}} + y_{n+\frac{1}{3}} - h^2\left[\frac{1}{108}f_{n+\frac{1}{3}} + \frac{5}{54}f_{n+\frac{2}{3}} + \frac{1}{108}f_{n+1}\right] = 0 \tag{4.3.1}$$

Expanding (4.3.1) in Taylor series in the form

$$\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j)} - 2\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j h^j}{j!}y_n^{(j)} + \sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^j h^j}{j!}y_n^{(j)} - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{1}{108}\left(\frac{1}{3}\right)^j + \frac{5}{54}\left(\frac{2}{3}\right)^j + \frac{1}{108}\right] \tag{4.3.2}$$

and collecting terms in powers of h and y leads to the following:

$$\begin{aligned}
c_0 &= 1-2+1=0, \quad c_1 = 1-2\left(\frac{2}{3}\right)+\frac{1}{3}=0 \\
c_2 &= \frac{1}{2!}-2\left(\frac{2}{3}\right)^2\frac{1}{2!}+\left(\frac{1}{3}\right)^2\frac{1}{2!}-\left[\frac{1}{108}+\frac{5}{54}+\frac{1}{108}\right]=0 \\
c_3 &= \frac{1}{3!}-2\left(\frac{2}{3}\right)^3\frac{1}{3!}+\left(\frac{1}{3}\right)^3\frac{1}{3!}-\left[\frac{1}{108}\left(\frac{2}{3}\right)+\frac{5}{54}\left(\frac{1}{3}\right)-\frac{1}{108}\right]=0 \\
c_4 &= \frac{1}{4!}-2\left(\frac{2}{3}\right)^4\frac{1}{4!}+\left(\frac{1}{3}\right)^4\frac{1}{4!}-\frac{1}{2!}\left[\frac{1}{108}\left(\frac{1}{3}\right)^2+\frac{5}{54}\left(\frac{2}{3}\right)^2+\frac{1}{108}\right]=0 \\
c_5 &= \frac{1}{5!}-2\left(\frac{2}{3}\right)^5\frac{1}{5!}+\left(\frac{1}{3}\right)^5\frac{1}{5!}-\frac{1}{3!}\left[\frac{1}{108}\left(\frac{1}{3}\right)^3+\frac{5}{54}\left(\frac{2}{3}\right)^3+\frac{1}{108}\right]=0 \\
c_6 &= \frac{1}{6!}-2\left(\frac{2}{3}\right)^6\frac{1}{6!}+\left(\frac{1}{3}\right)^6\frac{1}{6!}-\frac{1}{4!}\left[\frac{1}{108}\left(\frac{1}{3}\right)^4+\frac{5}{54}\left(\frac{2}{3}\right)^4+\frac{1}{108}\right]=-\frac{1}{174960}
\end{aligned}$$

Hence, the main method (3.3.21a) is of order $p = 4$ with error constant $c_{p+2} = -5.7158 \times 10^{-6}$.

4.3.1.2 Order of the block method

Let (3.3.25) be expressed in the form

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{3}}\\ y_{n+\frac{2}{3}}\\ y_{n+1}\\ y'_{n+\frac{1}{3}}\\ y'_{n+\frac{2}{3}}\\ y'_{n+1} \end{bmatrix}-\begin{bmatrix} 1&\frac{h}{3}\\ 1&\frac{2h}{3}\\ 1&h\\ 0&1\\ 0&1\\ 0&1 \end{bmatrix}\begin{bmatrix} y_n\\ y'_n \end{bmatrix}-\begin{bmatrix} \frac{97}{3240}h^2\\ \frac{28}{405}h^2\\ \frac{13}{120}h^2\\ \frac{1}{8}h\\ \frac{1}{9}h\\ \frac{1}{8}h \end{bmatrix}[f_n]$$

$$
-\begin{bmatrix}
\frac{19}{540}h^2 & \frac{-13}{1080}h^2 & \frac{1}{405}h^2 \\
\frac{22}{135}h^2 & \frac{-2}{135}h^2 & \frac{2}{405}h^2 \\
\frac{3}{10}h^2 & \frac{3}{40}h^2 & \frac{1}{60}h^2 \\
\frac{19}{72}h & \frac{5}{72}h & \frac{1}{72}h \\
\frac{4}{9}h & \frac{1}{9}h & 0 \\
\frac{3}{8}h & \frac{3}{8}h & \frac{1}{8}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{3}} \\
f_{n+\frac{2}{3}} \\
f_{n+1}
\end{bmatrix} \tag{4.3.3}
$$

Expanding (4.3.3) in Taylor series leads to the equation of the form

$$
\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{3}hy_n' - \frac{97}{3240}h^2 y_n^{(2)} \\
\quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} y_n^{(j+2)} \left[\frac{19}{540}\left(\frac{1}{3}\right)^j - \frac{13}{1080}\left(\frac{1}{3}\right)^j + \frac{1}{405}\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{2}{3}hy_n' - \frac{28}{405}h^2 y_n^{(2)} \\
\quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{22}{135}\left(\frac{1}{3}\right)^j - \frac{2}{135}\left(\frac{2}{3}\right)^j + \frac{2}{405}\right] \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j)} - y_n - hy_n' - \frac{13}{120}h^2 y_n^{(2)} \\
\quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{3}{10}\left(\frac{1}{3}\right)^j + \frac{3}{40}\left(\frac{2}{3}\right)^j + \frac{1}{60}\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{1}{8}hy_n^{(2)} \\
\quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{19}{72}\left(\frac{1}{3}\right)^j - \frac{5}{72}\left(\frac{2}{3}\right)^j + \frac{1}{72}\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{1}{9}hy_n^{(2)} \\
\quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{4}{9}\left(\frac{1}{3}\right)^j + \frac{1}{9}\left(\frac{2}{3}\right)^j\right] \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{1}{8}hy_n^{(2)} \\
\quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{3}{8}\left(\frac{1}{3}\right)^j + \frac{3}{8}\left(\frac{2}{3}\right)^j + \frac{1}{8}\right]
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} \tag{4.3.4}
$$

and collecting terms in powers of h and y leads to the following:

$$\bar{c}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{c}_1 = \begin{bmatrix} \frac{1}{3}-\frac{1}{3} \\ \frac{2}{3}-\frac{2}{3} \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_2 = \begin{bmatrix} \left(\frac{1}{3}\right)^2 \frac{1}{2!} - \frac{97}{3240} - \left[\frac{19}{540} - \frac{13}{1080}\frac{1}{405}\right] \\ \left(\frac{2}{3}\right)^2 \frac{1}{2!} - \frac{28}{405} - \left[\frac{22}{135} - \frac{2}{135} + \frac{2}{405}\right] \\ \frac{1}{2!} - \frac{13}{120} - \left[\frac{3}{10} + \frac{3}{40} + \frac{1}{60}\right] \\ \frac{1}{3} - \frac{1}{8} - \left[\frac{19}{72} - \frac{5}{72} + \frac{1}{72}\right] \\ \frac{2}{3} - \frac{1}{9} - \left[\frac{4}{9} + \frac{1}{9}\right] \\ 1 - \frac{1}{8} - \left[\frac{3}{8} + \frac{3}{8} + \frac{1}{8}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_3 = \begin{bmatrix} \left(\frac{1}{3}\right)^3 \frac{1}{3!} - \left[\frac{19}{540}\left(\frac{1}{3}\right) - \frac{13}{1080}\left(\frac{2}{3}\right) + \frac{1}{405}\right] \\ \left(\frac{2}{3}\right)^3 \frac{1}{3!} - \left[\frac{22}{135}\left(\frac{1}{3}\right) - \frac{2}{135}\left(\frac{2}{3}\right) + \frac{2}{405}\right] \\ \frac{1}{3!} - \left[\frac{3}{10}\left(\frac{1}{3}\right) + \frac{3}{40}\left(\frac{2}{3}\right) + \frac{1}{60}\right] \\ \left(\frac{1}{3}\right)^2 \frac{1}{2!} - \left[\frac{19}{72}\left(\frac{1}{3}\right) - \frac{5}{72}\left(\frac{2}{3}\right) + \frac{1}{72}\right] \\ \left(\frac{2}{3}\right)^2 \frac{1}{2!} - \left[\frac{4}{9}\left(\frac{1}{3}\right) + \frac{1}{9}\left(\frac{2}{3}\right)\right] \\ \frac{1}{2!} - \left[\frac{3}{8}\left(\frac{1}{3}\right) + \frac{3}{8}\left(\frac{2}{3}\right) + \frac{1}{8}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_4 = \begin{bmatrix} \left(\frac{1}{3}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{19}{540}\left(\frac{1}{3}\right)^2 - \frac{13}{1080}\left(\frac{2}{3}\right)^2 + \frac{1}{405}\right] \\ \left(\frac{2}{3}\right)^3 \frac{1}{4!} - \frac{1}{2!}\left[\frac{22}{135}\left(\frac{1}{3}\right)^2 - \frac{2}{135}\left(\frac{2}{3}\right)^2 + \frac{2}{405}\right] \\ \frac{1}{4!} - \frac{1}{2!}\left[\frac{3}{10}\left(\frac{1}{3}\right)^2 + \frac{3}{40}\left(\frac{2}{3}\right)^2 + \frac{1}{60}\right] \\ \left(\frac{1}{3}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{19}{72}\left(\frac{1}{3}\right)^2 - \frac{5}{72}\left(\frac{2}{3}\right)^2 + \frac{1}{72}\right] \\ \left(\frac{2}{3}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{4}{9}\left(\frac{1}{3}\right)^2 + \frac{1}{9}\left(\frac{2}{3}\right)^2\right] \\ \frac{1}{3!} - \frac{1}{2!}\left[\frac{3}{8}\left(\frac{1}{3}\right)^2 + \frac{3}{8}\left(\frac{2}{3}\right)^2 + \frac{1}{8}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_5 = \begin{bmatrix} \left(\frac{1}{3}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{19}{540}\left(\frac{1}{3}\right)^3 - \frac{13}{1080}\left(\frac{2}{3}\right)^3 + \frac{1}{405}\right] \\ \left(\frac{2}{3}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{22}{135}\left(\frac{1}{3}\right)^2 - \frac{2}{135}\left(\frac{2}{3}\right)^3 + \frac{2}{405}\right] \\ \frac{1}{5!} - \frac{1}{3!}\left[\frac{3}{10}\left(\frac{1}{3}\right)^3 + \frac{3}{40}\left(\frac{2}{3}\right)^3 + \frac{1}{60}\right] \\ \left(\frac{1}{3}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{19}{72}\left(\frac{1}{3}\right)^3 - \frac{5}{72}\left(\frac{2}{3}\right)^3 + \frac{1}{72}\right] \\ \left(\frac{2}{3}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{4}{9}\left(\frac{1}{3}\right)^3 - \frac{1}{9}\left(\frac{2}{3}\right)^3\right] \\ \frac{1}{4!} - \frac{1}{3!}\left[\frac{3}{8}\left(\frac{1}{3}\right)^3 + \frac{3}{8}\left(\frac{2}{3}\right)^3 + \frac{1}{8}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_6 = \begin{bmatrix} \left(\frac{1}{3}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{19}{540}\left(\frac{1}{3}\right)^4 - \frac{13}{1080}\left(\frac{2}{3}\right)^4 + \frac{1}{405}\right] \\ \left(\frac{2}{3}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{22}{135}\left(\frac{1}{3}\right)^4 - \frac{2}{135}\left(\frac{2}{3}\right)^4 + \frac{2}{405}\right] \\ \frac{1}{6!} - \frac{1}{4!}\left[\frac{3}{10}\left(\frac{1}{3}\right)^4 + \frac{3}{40}\left(\frac{2}{3}\right)^4 + \frac{1}{60}\right] \\ \left(\frac{1}{3}\right) 5\frac{1}{5!} - \frac{1}{4!}\left[\frac{19}{72}\left(\frac{1}{3}\right)^4 - \frac{5}{72}\left(\frac{2}{3}\right) 4 + \frac{1}{72}\right] \\ \left(\frac{2}{3}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{4}{9}\left(\frac{1}{3}\right)^4 + \frac{1}{9}\left(\frac{2}{3}\right)^4\right] \\ \frac{1}{5!} - \frac{1}{4!}\left[\frac{3}{8}\left(\frac{1}{3}\right)^4 + \frac{3}{8}\left(\frac{2}{3}\right)^4 + \frac{1}{8}\right] \end{bmatrix} = \begin{bmatrix} -\frac{7}{349920} \\ -\frac{1}{21870} \\ -\frac{1}{12960} \\ -\frac{19}{174960} \\ -\frac{1}{21870} \\ -\frac{1}{6480} \end{bmatrix}$$

Thus the block method (3.3.25) has order $p = (4, 4, 4, 4)^T$ with the error constant $\bar{c}_{p+2} = \left(\frac{-7}{349920},\ -\frac{1}{21870},\ -\frac{1}{21960},\ -\frac{19}{174960},\ -\frac{1}{21870},\ -\frac{1}{6480}\right)^T$.

4.3.2 Consistency of the Method

The block method (3.3.25) has order $p = (4, 4, 4, 4)^T \geq 1$, therefore it is consistent by condition (i) of definition (4.1.6).

Following the consistency of the block method (3.3.25), the consistency of the main method (3.3.21a) is shown by conditions (i) - (iv) of definition (4.1.6) below. However, consider the first and second characteristic polynomials of method (3.3.21a) given by

$$\rho(z) = z - 2z^{2/3} + z^{1/3} \tag{4.3.5}$$

and

$$\sigma(z) = \frac{z + 10z^{2/3} + z^{1/3}}{108} \tag{4.3.6}$$

Now by definition (4.1.6),

Condition (i)

The main method (3.3.21a) has been shown to have order

$$p = 4 \geq 1$$

Condition (ii)

$$\alpha_{1/3} = 1, \quad \alpha_{2/3} = -2 \text{ and } \alpha_1 = 1$$

$$\sum \alpha_j = 1 \quad - 2 + 1 = 1, \quad j = \frac{1}{3}, \frac{2}{3}, 1$$

Condition (iii)

$$\rho'(z) = \frac{1}{3} z^{-2/3} \left(3z^{2/3} - 4z^{1/3} + 1\right)$$

$$\rho'(1) = \frac{1}{3}(3 - 4 + 1) = 0$$

$$\rho(1) = 1 - 2 + 1 = 0$$

$$\Rightarrow \quad \rho(1) = \rho'(1) = 0$$

Condition (iv)

$$\rho''(z) = \frac{2}{9}z^{-5/3}\left(2z^{1/3} - 1\right)$$

$$\Rightarrow \quad \rho''(1) = \frac{2}{9}(2-1) = \frac{2}{9}$$

$$\sigma(1) = \frac{1+10+1}{108} = \frac{12}{108} = \frac{3}{27} = \frac{1}{9}$$

$$2!\sigma(1) = 2\left(\frac{1}{9}\right) = \frac{2}{9}$$

$$\Rightarrow \quad \rho''(1) = 2!\sigma(1)$$

Therefore the method (3.3.21a) is consistent.

4.3.3 Zero Stability of the Method

4.3.3.1 Zero stability of the Block Method (3.3.25)

From (3.3.25) using definitions in (3.2.12) as $h \to 0$ we have

$$\rho(z) = \det\left[z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right]$$

$$= \det\begin{bmatrix} z & 0 & -1 & 0 & 0 & 0 \\ 0 & z & -1 & 0 & 0 & 0 \\ 0 & 0 & z-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & r & 0 & z \end{bmatrix} = z^5(z-1)$$

Solving for z in

$$z^5(z-1)=0$$

gives $z=0$ or $z=1$.

Therefore, the block method (3.3.25) is stable since $|z|=1$ is simple.

Next, the zero stability of the method (3.3.21a) is considered.

4.3.3.2 Zero stability of the main method (3.3.21a)

The first characteristic polynomial of equation (3.3.21a) is given by (4.3.5) as

$$\rho(r)=z-2z^{2/3}+z^{1/2}$$

equating to zero and solving for r

$$z^{1/3}\left(z^{1/2}-1\right)=0$$

$$\Rightarrow \quad z=0 \quad \text{or} \quad z=1$$

Since $|z|=1$ is simple hence, the method is zero stable in the limit as $h\to 0$ by definition (4.1.7) and by the stability of the block method (3.3.25).

4.3.4 Convergence

Applying Theorem 4.1.1, the method (3.3.21a) is convergent since it satisfies the necessary and sufficient conditions of zero stability and consistency.

4.3.5 Region of Absolute Stability of the One Step Method with Two Offstep Points

The first and second characteristic polynomial of the method (3.3.21a) is given by equations (4.3.5) and (4.3.6) respectively. Therefore, the boundary of the region of absolute stability is given as follows:

$$\bar{h}(z)=\frac{\rho(z)}{\sigma(z)}=\frac{108(z-2z^{2/3}+z^{1/3})}{z+10z^{2/3}+z^{1/3}} \tag{4.3.7}$$

Let $z = e^{i\theta}$, therefore, (4.3.7) becomes

$$\bar{h}(\theta) = \frac{108(e^{i\theta} - 2e^{i\frac{2}{3}\theta} + e^{i\frac{1}{3}\theta})}{e^{i\theta} + 10e^{i\frac{2}{3}\theta} + e^{i\frac{1}{3}\theta}} \tag{4.3.8}$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$, (4.3.8) reduces after some manipulations to

$$\bar{h}(\theta) = \frac{1728\cos\frac{1}{3}\theta + 216\cos\frac{2}{3}\theta - 1944}{40\cos\frac{1}{3}\theta + 2\cos\frac{2}{3}\theta + 102} \tag{4.3.9}$$

Evaluating (4.3.9) at intervals of 30^0 gives the following results;

Table 2. The boundaries of the region of absolute stability of the one step 2 offstep points method.

θ	0	30^0	60^0	90^0	120^0	150^0	180^0
$\bar{h}(\theta)$	0	-0.27	-1.10	-2.47	-4.38	-6.84	-9.82

From table 2, the interval of absolute stability is $(-9.82, 0)$. The region of absolute stability is shown in figure 4.2.

Remark The locus is symmetric about the x-axis as $x(-\theta) = x(\theta)$

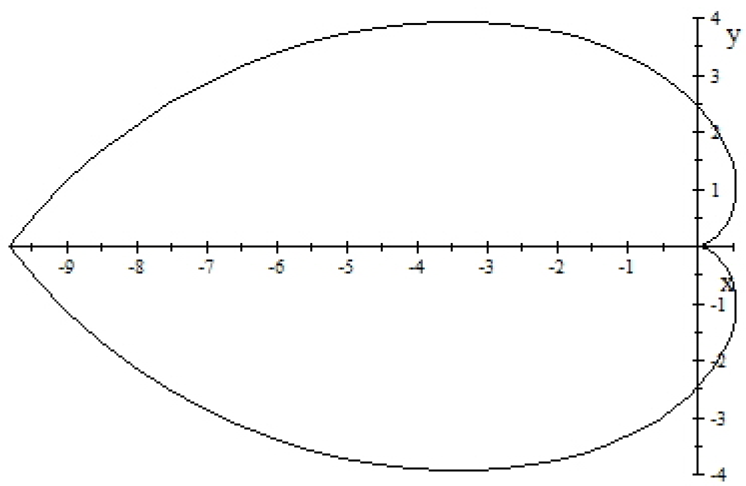

Figure 4.2: Region of Absolute Stability of the Continuous One-Step Method with Two Offstep Points

4.4 Analysis of the One Step Method with Three Offstep Points

In this section, order, error constant, consistency and zero stability of the main method (3.3.34c) and the associated block method (3.3.38) are obtained. The region of absolute stability of the method (3.3.34c) is also obtained.

4.4.1 Order and Error Constant

4.4.1.1 Order and error constant of the main method (3.3.34c)

Let (3.3.34c) be written in the form:

$$y_{n+1} - 2y_{n+\frac{3}{4}} + y_{n+\frac{1}{2}} - h^2\left[\frac{19}{3840}f_{n+1} + \frac{17}{320}f_{n+\frac{3}{4}} + \frac{7}{1920}f_{n+\frac{1}{2}} + \frac{1}{960}f_{n+\frac{1}{4}} - \frac{1}{3840}f_n\right] = 0 \tag{4.4.1}$$

Expanding (4.4.1) in Taylor series in the form

$$\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j)} - 2\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^j h^j}{j!}y_n^{(j)} + \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j h^j}{j!}y_n^{(j)} - \frac{1}{3840}h^2 y_n^{(2)} - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{1}{960}\left(\frac{1}{4}\right)^j + \frac{7}{1920}\left(\frac{1}{2}\right)^j + \frac{17}{320}\left(\frac{3}{4}\right)^j + \frac{19}{3840}\right] \tag{4.4.2}$$

and collecting terms in powers of h and y leads to the following:

$$
\begin{aligned}
c_0 &= 1-2-1=0\\
c_1 &= 1-2\left(\frac{3}{4}\right)+\frac{1}{2}=0\\
c_2 &= \frac{1}{2!}-2\left(\frac{3}{4}\right)^2\frac{1}{2!}+\left(\frac{1}{2}\right)^2\frac{1}{2!}-\frac{1}{3840}-\left[\frac{1}{960}+\frac{7}{1920}+\frac{17}{320}+\frac{19}{3840}\right]=0\\
c_3 &= \frac{1}{3!}-2\left(\frac{3}{4}\right)^3\frac{1}{3!}+\left(\frac{1}{2}\right)^3\frac{1}{3!}-\left[\frac{1}{960}\left(\frac{1}{4}\right)+\frac{7}{1920}\left(\frac{1}{2}\right)+\frac{17}{320}\left(\frac{3}{4}\right)\right.\\
&\quad\left.+\frac{19}{3840}\right]=0\\
c_4 &= \frac{1}{4!}-2\left(\frac{3}{4}\right)^4\frac{1}{4!}+\left(\frac{1}{2}\right)^4\frac{1}{4!}-\frac{1}{2!}\left[\frac{1}{960}\left(\frac{1}{4}\right)^2+\frac{7}{1920}\left(\frac{1}{2}\right)^2+\frac{17}{320}\left(\frac{3}{4}\right)^2\right.\\
&\quad\left.+\frac{19}{3840}\right]=0\\
c_5 &= \frac{1}{5!}-2\left(\frac{3}{4}\right)^5\frac{1}{5!}+\left(\frac{1}{2}\right)^5\frac{1}{5!}-\frac{1}{3!}\left[\frac{1}{960}\left(\frac{1}{4}\right)^3+\frac{7}{1920}\left(\frac{1}{2}\right)^3+\frac{17}{320}\left(\frac{3}{4}\right)^3\right.\\
&\quad\left.+\frac{19}{3840}\right]=0
\end{aligned}
$$

$$
\begin{aligned}
c_6 &= \frac{1}{6!}-2\left(\frac{3}{4}\right)^6\frac{1}{6!}+\left(\frac{1}{2}\right)^6\frac{1}{6!}-\frac{1}{4!}\left[\frac{1}{960}\left(\frac{1}{4}\right)^4+\frac{7}{1920}\left(\frac{1}{2}\right)^4+\frac{17}{320}\left(\frac{3}{4}\right)^4\right.\\
&\quad\left.+\frac{19}{3840}\right]=0\\
c_7 &= \frac{1}{7!}-2\left(\frac{3}{4}\right)^7\frac{1}{7!}+\left(\frac{1}{2}\right)^7\frac{1}{7!}-\frac{1}{5!}\left[\frac{1}{960}\left(\frac{1}{4}\right)^5+\frac{7}{1920}\left(\frac{1}{2}\right)^5+\frac{17}{320}\left(\frac{3}{4}\right)^5\right.\\
&\quad\left.+\frac{19}{3840}\right]=\frac{1}{3932160}
\end{aligned}
$$

Hence, the method (3.3.34c) is of order $p = 5$ with error constant $c_{p+2} = 2.5431\times10^{-7}$.

4.4.1.2 Order of the block method (3.3.38)

Let (3.3.38) be expressed in the form

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{4}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{3}{4}}\\ y_{n+1}\\ y'_{n+\frac{1}{4}}\\ y'_{n+\frac{1}{2}}\\ y'_{n+\frac{3}{4}}\\ y'_{n+1}
\end{bmatrix}
-
\begin{bmatrix}
1 & \frac{1}{4}h\\
1 & \frac{1}{2}h\\
1 & \frac{3}{4}h\\
1 & h\\
0 & 1\\
0 & 1\\
0 & 1\\
0 & 1
\end{bmatrix}
\begin{bmatrix}
y_n\\ y'_n
\end{bmatrix}
-
\begin{bmatrix}
\frac{367}{23040}h^2\\
\frac{53}{1440}h^2\\
\frac{147}{2560}h^2\\
\frac{7}{90}h^2\\
\frac{251}{2880}h\\
\frac{29}{360}h^2\\
\frac{27}{320}h\\
\frac{7}{90}h
\end{bmatrix}
[f_n]
$$

$$
-\begin{bmatrix}
\frac{3}{128}h^2 & -\frac{47}{3840}h^2 & \frac{29}{5760}h^2 & -\frac{7}{7680}h^2\\
\frac{1}{10}h^2 & -\frac{1}{48}h^2 & \frac{1}{90}h^2 & -\frac{1}{480}h^2\\
\frac{117}{640}h^2 & \frac{27}{1280}h^2 & \frac{3}{128}h^2 & -\frac{9}{2560}h^2\\
\frac{4}{15}h^2 & \frac{1}{15}h^2 & \frac{4}{45}h^2 & 0\\
\frac{323}{1440}h & -\frac{11}{120}h & \frac{53}{1440}h & -\frac{19}{2880}h\\
\frac{31}{90}h & \frac{1}{15}h & \frac{1}{90}h & -\frac{1}{360}h\\
\frac{51}{160}h & \frac{9}{40}h & \frac{21}{160}h & -\frac{3}{320}h\\
\frac{16}{45}h & \frac{2}{15}h & \frac{16}{45}h & \frac{7}{90}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{4}}\\ f_{n+\frac{1}{2}}\\ f_{n+\frac{3}{4}}\\ f_{n+1}
\end{bmatrix}
\qquad (4.4.3)
$$

Expanding (4.4.3) in Taylor series in the form

$$\begin{bmatrix} \displaystyle\sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{4} h y_n^{(1)} - \frac{367}{23040} h^2 y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{3}{128} \left(\frac{1}{4}\right)^j - \frac{47}{3840} \left(\frac{1}{2}\right)^j + \frac{29}{5760} \left(\frac{3}{4}\right)^j - \frac{7}{7680} \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{2} h y_n^{(1)} - \frac{53}{1440} h^2 y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{1}{10} \left(\frac{1}{4}\right)^j - \frac{1}{48} \left(\frac{1}{2}\right)^j + \frac{1}{90} \left(\frac{3}{4}\right)^j - \frac{1}{480} \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{\left(\frac{3}{4}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{3}{4} h y_n^{(1)} - \frac{147}{2560} h^2 y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{117}{640} \left(\frac{1}{4}\right)^j + \frac{27}{1280} \left(\frac{1}{2}\right)^j + \frac{3}{128} \left(\frac{3}{4}\right)^j - \frac{9}{2560} \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j)} - y_n - h y_n^{(1)} - \frac{7}{90} h^2 y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{4}{15} \left(\frac{1}{4}\right)^j + \frac{1}{15} \left(\frac{1}{2}\right)^j + \frac{4}{45} \left(\frac{3}{4}\right)^j \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{251}{2880} h y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{323}{1440} \left(\frac{1}{4}\right)^j - \frac{11}{120} \left(\frac{1}{2}\right)^j + \frac{53}{1440} \left(\frac{3}{4}\right)^j - \frac{19}{2880} \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{29}{360} h y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{31}{90} \left(\frac{1}{4}\right)^j + \frac{1}{15} \left(\frac{1}{2}\right)^j + \frac{1}{90} \left(\frac{3}{4}\right)^j - \frac{1}{360} \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{\left(\frac{3}{4}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{27}{320} h y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{51}{160} \left(\frac{1}{4}\right)^j + \frac{9}{40} \left(\frac{1}{2}\right)^j + \frac{21}{160} \left(\frac{3}{4}\right)^j - \frac{3}{320} \right] \\ \displaystyle\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{7}{90} h y_n^{(2)} \\ \quad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{16}{45} \left(\frac{1}{4}\right)^j + \frac{2}{15} \left(\frac{1}{2}\right)^j + \frac{16}{45} \left(\frac{3}{4}\right)^j + \frac{7}{90} \right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.4.4)$$

and collecting terms in powers of h and y leads to the following results:

$$\bar{c}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{c}_1 = \begin{bmatrix} \frac{1}{4}-\frac{1}{4} \\ \frac{1}{2}-\frac{1}{2} \\ \frac{3}{4}-\frac{3}{4} \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_2 = \begin{bmatrix} \left(\frac{1}{4}\right)^2 \frac{1}{2!} - \frac{367}{23040} - \left[\frac{3}{128} - \frac{47}{3840} + \frac{29}{5760} - \frac{7}{7680}\right] \\ \left(\frac{1}{2}\right)^2 \frac{1}{2!} - \frac{53}{1440} - \left[\frac{1}{10} - \frac{1}{48} + \frac{1}{90} - \frac{1}{480}\right] \\ \left(\frac{3}{4}\right)^2 \frac{1}{2!} - \frac{147}{2560} - \left[\frac{117}{640} + \frac{27}{1280} + \frac{3}{128} - \frac{9}{2560}\right] \\ \frac{1}{2!} - \frac{7}{90} - \left[\frac{4}{15} + \frac{1}{15} + \frac{4}{45}\right] \\ \frac{1}{4} - \frac{251}{2880} - \left[\frac{323}{1440} - \frac{11}{120} + \frac{53}{1440} - \frac{19}{2880}\right] \\ \frac{1}{2} - \frac{29}{360} - \left[\frac{31}{90} + \frac{1}{15} + \frac{1}{90} - \frac{1}{360}\right] \\ \frac{3}{4} - \frac{27}{320} - \left[\frac{51}{160} + \frac{9}{40} + \frac{21}{160} - \frac{3}{320}\right] \\ 1 - \frac{7}{90} - \left[\frac{16}{95} + \frac{2}{15} + \frac{16}{45} + \frac{7}{90}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_3 = \begin{bmatrix} \left(\frac{1}{4}\right)^3 \frac{1}{3!} - \left[\frac{3}{128}\left(\frac{1}{4}\right) - \frac{47}{3840}\left(\frac{1}{4}\right) + \frac{29}{5760}\left(\frac{3}{4}\right) - \frac{7}{7680}\right] \\ \left(\frac{1}{2}\right)^3 \frac{1}{3!} - \left[\frac{1}{10}\left(\frac{1}{4}\right) - \frac{1}{48}\left(\frac{1}{2}\right) + \frac{1}{90}\left(\frac{3}{4}\right) - \frac{1}{480}\right] \\ \left(\frac{3}{4}\right)^3 \frac{1}{3!} - \left[\frac{117}{640}\left(\frac{1}{4}\right) + \frac{27}{1280}\left(\frac{1}{2}\right) + \frac{3}{128}\left(\frac{3}{4}\right) - \frac{9}{2560}\right] \\ \frac{1}{3!} - \left[\frac{4}{15}\left(\frac{1}{4}\right) + \frac{1}{15}\left(\frac{1}{2}\right) + \frac{4}{45}\left(\frac{3}{4}\right)\right] \\ \left(\frac{1}{4}\right)^2 \frac{1}{2!} - \left[\frac{323}{1440}\left(\frac{1}{4}\right) - \frac{11}{120}\left(\frac{1}{2}\right) + \frac{53}{1440}\left(\frac{3}{4}\right) - \frac{19}{2880}\right] \\ \left(\frac{1}{2}\right)^2 \frac{1}{2!} - \left[\frac{31}{90}\left(\frac{1}{4}\right) + \frac{1}{15}\left(\frac{1}{2}\right) + \frac{1}{90}\left(\frac{3}{4}\right) - \frac{1}{360}\right] \\ \left(\frac{3}{4}\right)^2 \frac{1}{2!} - \left[\frac{51}{160}\left(\frac{1}{4}\right) + \frac{9}{40}\left(\frac{1}{2}\right) + \frac{21}{160}\left(\frac{3}{4}\right) - \frac{3}{320}\right] \\ \frac{1}{2!} - \left[\frac{16}{45}\left(\frac{1}{4}\right) + \frac{2}{15}\left(\frac{1}{2}\right) + \frac{16}{45}\left(\frac{3}{4}\right) + \frac{7}{90}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_4 = \begin{bmatrix} \left(\frac{1}{4}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{3}{125}\left(\frac{1}{4}\right)^2 - \frac{47}{3840}\left(\frac{1}{2}\right)^2 + \frac{29}{5760}\left(\frac{3}{4}\right)^2 - \frac{7}{7680}\right] \\ \left(\frac{1}{2}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{1}{10}\left(\frac{1}{4}\right)^2 - \frac{1}{48}\left(\frac{1}{2}\right)^2 + \frac{1}{90}\left(\frac{3}{4}\right)^2 - \frac{1}{480}\right] \\ \left(\frac{3}{4}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{117}{640}\left(\frac{1}{4}\right)^2 + \frac{27}{1280}\left(\frac{1}{2}\right)^2 + \frac{3}{128}\left(\frac{3}{4}\right)^2 - \frac{9}{2560}\right] \\ \frac{1}{4!} - \frac{1}{2!}\left[\frac{4}{15}\left(\frac{1}{4}\right)^2 + \frac{1}{15}\left(\frac{1}{2}\right)^2 + \frac{4}{45}\left(\frac{3}{4}\right)^2\right] \\ \left(\frac{1}{4}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{323}{1440}\left(\frac{1}{4}\right)^2 - \frac{11}{120}\left(\frac{1}{2}\right)^2 + \frac{53}{1440}\left(\frac{3}{4}\right)^2 - \frac{19}{2880}\right] \\ \left(\frac{1}{2}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{31}{90}\left(\frac{1}{4}\right)^2 + \frac{1}{15}\left(\frac{1}{2}\right)^2 + \frac{1}{90}\left(\frac{3}{4}\right)^2 - \frac{1}{360}\right] \\ \left(\frac{3}{4}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{51}{160}\left(\frac{1}{4}\right)^2 + \frac{9}{40}\left(\frac{1}{2}\right)^2 + \frac{21}{160}\left(\frac{3}{4}\right)^2 - \frac{3}{320}\right] \\ \frac{1}{3!} - \frac{1}{2!}\left[\frac{16}{45}\left(\frac{1}{4}\right)^2 + \frac{2}{15}\left(\frac{1}{2}\right)^2 + \frac{16}{45}\left(\frac{3}{4}\right)^2 + \frac{7}{90}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_5 = \begin{bmatrix} \left(\frac{1}{4}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{3}{125}\left(\frac{1}{4}\right)^3 - \frac{47}{3840}\left(\frac{1}{2}\right)^3 + \frac{29}{5760}\left(\frac{3}{4}\right)^3 - \frac{7}{7680}\right] \\ \left(\frac{1}{2}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{1}{10}\left(\frac{1}{4}\right)^3 - \frac{1}{48}\left(\frac{1}{2}\right)^3 + \frac{1}{90}\left(\frac{3}{4}\right)^3 - \frac{1}{480}\right] \\ \left(\frac{3}{4}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{117}{640}\left(\frac{1}{4}\right)^3 + \frac{27}{1280}\left(\frac{1}{2}\right)^3 + \frac{3}{128}\left(\frac{3}{4}\right)^3 - \frac{9}{2560}\right] \\ \frac{1}{5!} - \frac{1}{3!}\left[\frac{4}{15}\left(\frac{1}{4}\right)^3 + \frac{1}{15}\left(\frac{1}{2}\right)^3 + \frac{4}{45}\left(\frac{3}{4}\right)^3\right] \\ \left(\frac{1}{4}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{323}{1440}\left(\frac{1}{4}\right)^3 - \frac{11}{120}\left(\frac{1}{2}\right)^3 - \frac{53}{1440}\left(\frac{3}{4}\right)^3 - \frac{19}{2880}\right] \\ \left(\frac{1}{2}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{31}{90}\left(\frac{1}{4}\right)^3 + \frac{1}{15}\left(\frac{1}{2}\right)^3 + \frac{1}{90}\left(\frac{3}{4}\right)^3 - \frac{1}{360}\right] \\ \left(\frac{3}{4}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{51}{160}\left(\frac{1}{4}\right)^3 + \frac{9}{40}\left(\frac{1}{2}\right)^3 + \frac{21}{160}\left(\frac{3}{4}\right)^3 - \frac{3}{320}\right] \\ \frac{1}{4!} - \frac{1}{3!}\left[\frac{16}{45}\left(\frac{1}{4}\right)^3 + \frac{2}{15}\left(\frac{1}{2}\right)^3 + \frac{16}{45}\left(\frac{3}{4}\right)^3 + \frac{7}{90}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_6 = \begin{bmatrix} \left(\frac{1}{4}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{3}{125}\left(\frac{1}{4}\right)^4 - \frac{47}{3840}\left(\frac{1}{2}\right)^4 + \frac{29}{5760}\left(\frac{3}{4}\right)^4 - \frac{7}{7680}\right] \\ \left(\frac{1}{2}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{1}{10}\left(\frac{1}{4}\right)^4 - \frac{1}{48}\left(\frac{1}{2}\right)^4 + \frac{1}{90}\left(\frac{3}{4}\right)^4 - \frac{1}{480}\right] \\ \left(\frac{3}{4}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{117}{640}\left(\frac{1}{4}\right)^4 + \frac{27}{1280}\left(\frac{1}{2}\right)^4 + \frac{3}{128}\left(\frac{3}{4}\right)^4 - \frac{9}{2560}\right] \\ \frac{1}{6!} - \frac{1}{4!}\left[\frac{4}{15}\left(\frac{1}{4}\right)^4 + \frac{1}{15}\left(\frac{1}{2}\right)^4 + \frac{4}{45}\left(\frac{3}{4}\right)^4\right] \\ \left(\frac{1}{4}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{323}{1440}\left(\frac{1}{4}\right)^4 - \frac{11}{120}\left(\frac{1}{2}\right)^4 - \frac{53}{1440}\left(\frac{3}{4}\right)^4 - \frac{19}{2880}\right] \\ \left(\frac{1}{2}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{31}{90}\left(\frac{1}{4}\right)^4 + \frac{1}{15}\left(\frac{1}{2}\right)^4 + \frac{1}{90}\left(\frac{3}{4}\right)^4 - \frac{1}{360}\right] \\ \left(\frac{3}{4}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{51}{160}\left(\frac{1}{4}\right)^4 + \frac{9}{40}\left(\frac{1}{2}\right)^4 + \frac{21}{160}\left(\frac{3}{4}\right)^4 - \frac{3}{320}\right] \\ \frac{1}{5!} - \frac{1}{4!}\left[\frac{16}{45}\left(\frac{1}{4}\right)^4 + \frac{2}{15}\left(\frac{1}{2}\right)^4 + \frac{16}{45}\left(\frac{3}{4}\right)^4 + \frac{7}{90}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_7 = \begin{bmatrix} \left(\frac{1}{4}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{3}{125}\left(\frac{1}{4}\right)^5 - \frac{47}{3840}\left(\frac{1}{2}\right)^5 + \frac{29}{5760}\left(\frac{3}{4}\right)^5 - \frac{7}{7680}\right] \\ \left(\frac{1}{2}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{1}{10}\left(\frac{1}{4}\right)^5 - \frac{1}{48}\left(\frac{1}{2}\right)^5 + \frac{1}{90}\left(\frac{3}{4}\right)^5 - \frac{1}{480}\right] \\ \left(\frac{3}{4}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{117}{640}\left(\frac{1}{4}\right)^5 + \frac{27}{1280}\left(\frac{1}{2}\right)^5 + \frac{3}{128}\left(\frac{3}{4}\right)^5 - \frac{9}{2560}\right] \\ \frac{1}{7!} - \frac{1}{5!}\left[\frac{4}{15}\left(\frac{1}{4}\right)^5 + \frac{1}{15}\left(\frac{1}{2}\right)^5 + \frac{4}{45}\left(\frac{3}{4}\right)^5\right] \\ \left(\frac{1}{4}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{323}{1440}\left(\frac{1}{4}\right)^5 - \frac{11}{120}\left(\frac{1}{2}\right)^5 - \frac{53}{1440}\left(\frac{3}{4}\right)^5 - \frac{19}{2880}\right] \\ \left(\frac{1}{2}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{31}{90}\left(\frac{1}{4}\right)^5 + \frac{1}{15}\left(\frac{1}{2}\right)^5 + \frac{1}{90}\left(\frac{3}{4}\right)^5 - \frac{1}{360}\right] \\ \left(\frac{3}{4}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{51}{160}\left(\frac{1}{4}\right)^5 + \frac{9}{40}\left(\frac{1}{2}\right)^5 + \frac{21}{160}\left(\frac{3}{4}\right)^5 - \frac{3}{320}\right] \\ \frac{1}{6!} - \frac{1}{5!}\left[\frac{16}{45}\left(\frac{1}{4}\right)^5 + \frac{2}{15}\left(\frac{1}{2}\right)^5 + \frac{16}{45}\left(\frac{3}{4}\right)^5 + \frac{7}{90}\right] \end{bmatrix} = \begin{bmatrix} \frac{26561}{41287680000} \\ \frac{1}{645120} \\ \frac{9}{3670016} \\ \frac{1}{322560} \\ \frac{123}{819200} \\ \frac{1}{368640} \\ \frac{3}{655360} \\ -\frac{7}{1350} \end{bmatrix}$$

Hence the block method (3.3.38) is of order $p = (5,5,5,5,5,5,5,5)^T$ and error constant $\bar{c}_{p+2} = \left(\frac{26561}{41287680000}, \frac{1}{645120}, \frac{9}{3670016}, \frac{1}{322560}, \frac{123}{819200}, \frac{1}{368640}, \frac{3}{655360}, -\frac{7}{1350}\right)^T$.

4.4.2 Consistency

The block method (3.3.38) has order $p = (5,5,5,5,5,5,5,5)^T > 1$ therefore by condition (i) of definition (4.1.6) it is consistent.

Following the consistency of the block method (3.3.38), the consistency of the method (3.3.34c) is shown as follows by conditions (i) - (iv) of definition (4.1.6). Consider the first and second characteristic p olynomials of the method (3.3.34c) are given by

$$\rho(z) = z - 2z^{3/4} + z^{1/2} \tag{4.4.5}$$

and

$$\sigma(z) = \frac{19z + 204z^{3/4} + 14z^{1/2} + 4z^{1/4} - 1}{3840} \tag{4.4.6}$$

The conditions in definition (4.1.6) are satisfied as follows;

Condition (i)

This condition is satisfied since the method (3.3.34c) has order$p = 5 > 1$

Condition (ii)

$$\alpha_{\frac{1}{2}} = 1, \quad \alpha_{\frac{3}{4}} = -2 \text{ and } \alpha_1 = 1$$

$$\therefore \sum_j \alpha_j = 1 - 2 + 1 = 0, \quad j = \frac{1}{2}, \frac{3}{4}, 1$$

Condition (iii)

$$\begin{aligned} \rho'(z) &= \frac{1}{2\sqrt{z}}(2\sqrt{z} - 3\sqrt[4]{z} + 1) \\ \rho'(1) &= \frac{2 - 3 + 1}{2} = 0 \end{aligned}$$

Now by (4.4.5)

$$\begin{aligned} \rho(1) &= 1 - 2 + 1 = 0 \\ \therefore \rho(1) &= \rho'(1) = 0 \end{aligned}$$

Condition (iv)

$$\rho''(z) = \frac{1}{8(\sqrt{z})^3}(3\sqrt[4]{z} - 2)$$

Thus

$$\rho''(1) = \frac{1}{8}(3 - 2) = \frac{1}{8}$$

Also, by (4.4.6)

$$\begin{aligned} \sigma(1) &= \frac{19 + 204 + 14 + 4 - 1}{3840} = \frac{1}{16} \\ 2!\sigma(1) &= 2\left(\frac{1}{16}\right) = \frac{1}{8} \\ \therefore \rho''(1) &= 2!\sigma(1). \end{aligned}$$

Therefore by definition (4.1.6), the method (3.3.34c) is consistent.

4.4.3 Zero Stability

4.4.3.1 Zero stability of the block method (3.3.38)

From (3.3.38) and using definition in (3.2.12) in the limit as $h \to 0$. The first characteristic polynomial of the block method is given by

$$\rho(z) = \det \left[z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] - \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(r) = \det \begin{bmatrix} z & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{bmatrix}$$

i.e.

$$\rho(z) = z^7(z-1) \tag{4.4.7}$$

Equating (4.4.7) to zero and solving for r gives

$$z = 0 \quad \text{or} \quad r = 1$$

Since no root has modulus greater than one and $|r| = 1$ is simple, the block method is zero stable in the limit as $h \to 0$ by definition (4.1.7).

4.4.3.2 Zero stability of the main method (3.3.34c)

From (3.3.34c), the first characteristic polynomial is given by

$$\rho(z) = z - 2z^{3/4} + z^{1/2}$$

Equating the above equation to zero and solving for z gives $z = 0$ or $z = 1$.

Since no root of the polynomial has modulus greater than 1 and $|z| = 1$ is simple, it follows from definition (4.1.7) that the method (3.3.34c) is zero stable.

4.4.4 Convergence

Following Theorem 4.1.1, the method (3.3.34c) is convergent since it is consistent and zero stable.

4.4.5 Region of Absolute stability of the One Step Method with Three Offstep Points

The first and second characteristic polynomials of the method (3.3.34c) have been given as (4.4.5) and (4.4.6) respectively. Now, by the boundary locus method, the boundary of the region of absolute stability is given by (4.1.14) as

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} = \frac{3840(e^{i\theta} - 2e^{i\frac{3}{4}\theta} + e^{i\frac{1}{2}\theta})}{19e^{i\theta} + 204e^{i\frac{3}{4}\theta} + 14e^{i\frac{1}{2}\theta} + 4e^{i\frac{1}{4}\theta} - 1} \tag{4.4.8}$$

where $e^{i\theta} = r$ is the value of the root of the stability polynomial (4.1.10).

Since $e^{i\theta} = \cos\theta + i\sin\theta$, (4.4.8) reduces after some manipulations to

$$\bar{h}(\theta) = \frac{92160\cos\frac{1}{2}\theta + 1328640\cos\frac{1}{4}\theta + 23040\cos\frac{3}{4}\theta - 3840\cos\theta - 1440000}{2136\cos\frac{1}{2}\theta + 13568\cos\frac{1}{4}\theta - 256\cos\frac{3}{4}\theta - 38\cos\theta + 42190} \tag{4.4.9}$$

Evaluating (4.4.9) at intervals of 30^0 gives the following results in table 3

Table 3. The boundaries of the region of absolute stability of the one step 3 offstep points method.

$x(\theta)$	0	30^0	60^0	90^0	120^0	150^0	180^0
$\bar{h}(\theta)$	0	-0.28	-1.10	-2.54	-4.39	-6.85	-9.86

From table 3, the interval of absolute stability is (-9.86,0). The region of absolute stability is figure 4.3.

Remark

The locus is symmetric about the x-axis as $x(-\theta) = x(\theta)$.

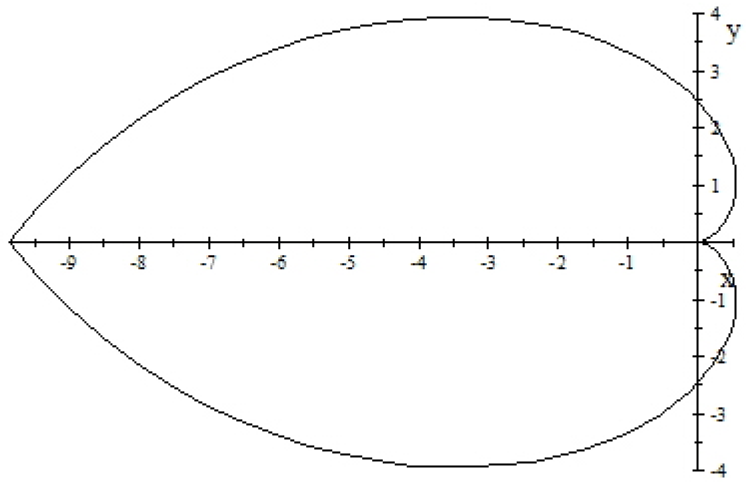

Figure 4.3: Region of Absolute Stability of the Continuous One Step Method with Three Offstep Points

4.5 Analysis of the One Step Method with Four Offstep Points

In this section, the basic properties such as order, error constant, consistency and zero stability are obtained for the method (3.3.47d) and the associated block method (3.3.51). The region of absolute stability of the method (3.3.47d) is also obtained.

4.5.1 Order and Error Constant

4.5.1.1 Order and Error Constant of the Main Method (3.3.47d)

Let (3.3.57d) be written in the form

$$y_{n+1}-2y_{n+\frac{4}{5}}+y_{n+\frac{3}{5}}-\frac{h^2}{6000}\left[18f_{n+1}+209f_{n+\frac{4}{5}}+4f_{n+\frac{3}{5}}+14f_{n+\frac{2}{5}}-6f_{n+\frac{1}{5}}+f_n\right] \quad (4.5.1)$$

Expanding (4.5.1) in Taylor series in the form

$$\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^{(j)}-2\sum_{j=0}^{\infty}\frac{\left(\frac{4}{5}\right)^j h^j}{j!}y_n^{(j)}+\sum_{j=0}^{\infty}\frac{\left(\frac{3}{5}\right)^j h^j}{j!}y_n^{(j)}-\frac{1}{6000}h^2y_n^{(2)}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{(j+2)}\left[\frac{-1}{1000}\left(\frac{1}{5}\right)^j\right.$$

$$\left.+\frac{7}{3000}\left(\frac{2}{5}\right)^j+\frac{1}{1500}\left(\frac{3}{5}\right)^j+\frac{209}{6000}\left(\frac{4}{5}\right)^j+\frac{3}{1000}\right] \quad (4.5.2)$$

and collecting terms in powers of h and y leads to

$$\begin{aligned}
c_0 &= 1-2+1=0\\
c_1 &= 1-\frac{8}{5}+\frac{3}{5}=0\\
c_2 &= \frac{1}{2!}-2\left(\frac{4}{5}\right)^2\frac{1}{2!}+\left(\frac{3}{5}\right)^2\frac{1}{2!}-\frac{1}{6000}-\left[-\frac{1}{1000}+\frac{7}{3000}+\frac{1}{1500}+\frac{209}{6000}+\frac{3}{1000}\right]=0\\
c_3 &= \frac{1}{3!}-2\left(\frac{4}{5}\right)^3\frac{1}{3!}+\left(\frac{3}{5}\right)^3\frac{1}{3!}-\left[-\frac{1}{1000}\left(\frac{1}{5}\right)+\frac{7}{3000}\left(\frac{2}{5}\right)+\frac{1}{1500}\left(\frac{3}{5}\right)\right.\\
&\quad\left.+\frac{209}{6000}\left(\frac{4}{5}\right)+\frac{3}{1000}\right]=0
\end{aligned}$$

$$c_4 = \frac{1}{4!} - 2\left(\frac{4}{5}\right)^4 \frac{1}{4!} + \left(\frac{3}{5}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[-\frac{1}{1000}\left(\frac{1}{5}\right)^2 + \frac{7}{3000}\left(\frac{2}{5}\right)^2 + \frac{1}{1500}\left(\frac{3}{5}\right)^2 + \frac{209}{6000}\left(\frac{4}{5}\right)^2 + \frac{3}{1000}\right] = 0$$

$$c_5 = \frac{1}{5!} - 2\left(\frac{4}{5}\right)^5 \frac{1}{5!} + \left(\frac{3}{5}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[-\frac{1}{1000}\left(\frac{1}{5}\right)^3 + \frac{7}{3000}\left(\frac{2}{5}\right)^3 + \frac{1}{1500}\left(\frac{3}{5}\right)^3 + \frac{209}{6000}\left(\frac{4}{5}\right)^3 + \frac{3}{1000}\right] = 0$$

$$c_6 = \frac{1}{6!} - 2\left(\frac{4}{5}\right)^6 \frac{1}{6!} + \left(\frac{3}{5}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[-\frac{1}{1000}\left(\frac{1}{5}\right)^4 + \frac{7}{3000}\left(\frac{2}{5}\right)^4 + \frac{1}{1500}\left(\frac{3}{5}\right)^4 + \frac{209}{6000}\left(\frac{4}{5}\right)^4 + \frac{3}{1000}\right] = 0$$

$$c_7 = \frac{1}{7!} - 2\left(\frac{4}{5}\right)^7 \frac{1}{7!} + \left(\frac{3}{5}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[-\frac{1}{1000}\left(\frac{1}{5}\right)^5 + \frac{7}{3000}\left(\frac{2}{5}\right)^5 + \frac{1}{1500}\left(\frac{3}{5}\right)^5 + \frac{209}{6000}\left(\frac{4}{5}\right)^5 + \frac{3}{1000}\right] = 0$$

$$c_8 = \frac{1}{8!} - 2\left(\frac{4}{5}\right)^8 \frac{1}{8!} + \left(\frac{3}{5}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[-\frac{1}{1000}\left(\frac{1}{5}\right)^6 + \frac{7}{3000}\left(\frac{2}{5}\right)^6 + \frac{1}{1500}\left(\frac{3}{5}\right)^6 + \frac{209}{6000}\left(\frac{4}{5}\right)^6 + \frac{3}{1000}\right] = -\frac{221}{2362500000}$$

Hence, the method (3.3.47d) is of order $p = 6$ with error constant

$c_{p+2} = -9.3545 \times 10^{-9}$.

4.5.1.2 Order of the block method (3.3.51)

Let (3.3.51) be expressed in the form

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{5}}\\ y_{n+\frac{2}{5}}\\ y_{n+\frac{3}{5}}\\ y_{n+\frac{4}{5}}\\ y_{n+1}\\ y'_{n+\frac{1}{5}}\\ y'_{n+\frac{2}{5}}\\ y'_{n+\frac{3}{5}}\\ y'_{n+\frac{4}{5}}\\ y'_{n+1}
\end{bmatrix}
-
\begin{bmatrix}
1&\frac{1}{5}h\\
1&\frac{2}{5}h\\
1&\frac{3}{5}h\\
1&\frac{4}{5}h\\
1&h\\
0&1\\
0&1\\
0&1\\
0&1\\
0&1
\end{bmatrix}
\begin{bmatrix}
y_n\\ y'_n
\end{bmatrix}
-
\begin{bmatrix}
\frac{1231}{126000}h^2\\
\frac{71}{3150}h^2\\
\frac{123}{3500}h^2\\
\frac{376}{7875}h^2\\
\frac{61}{1008}h^2\\
\frac{19}{288}h\\
\frac{15}{225}h\\
\frac{51}{800}h\\
\frac{14}{225}\\
\frac{19}{288}h
\end{bmatrix}
[f_n]
$$

$$
-\begin{bmatrix}
\frac{863}{50400}h^2 & -\frac{761}{63000}h^2 & \frac{941}{126000}h^2 & -\frac{341}{126000}h^2 & \frac{107}{252000}h^2\\
\frac{544}{7875}h^2 & -\frac{37}{1575}h^2 & \frac{136}{7875}h^2 & -\frac{101}{15750}h^2 & \frac{8}{7875}h^2\\
\frac{3501}{28000}h^2 & -\frac{9}{3500}h^2 & \frac{87}{2800}h^2 & -\frac{9}{875}h^2 & \frac{9}{5600}h^2\\
\frac{1424}{7875}h^2 & \frac{176}{7875}h^2 & \frac{608}{7875}h^2 & -\frac{16}{1575}h^2 & \frac{16}{7875}h^2\\
\frac{475}{2016}h^2 & \frac{25}{504}h^2 & \frac{125}{1008}h & \frac{25}{1008}h^2 & \frac{11}{2016}h^2\\
\frac{1427}{7200}h & -\frac{133}{1200}h & \frac{241}{3600}h & -\frac{173}{7200}h & \frac{3}{800}h\\
\frac{43}{150}h & \frac{7}{225}h & \frac{7}{225}h & -\frac{1}{75}h & \frac{1}{450}h\\
\frac{219}{800}h & \frac{57}{400}h & \frac{57}{400}h & -\frac{21}{800}h & \frac{3}{800}h\\
\frac{64}{225}h & \frac{8}{75}h & \frac{64}{225}h & \frac{14}{225}h & 0\\
\frac{25}{96}h & \frac{25}{144}h & \frac{25}{144}h & \frac{25}{96}h & \frac{19}{288}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{5}}\\ f_{n+\frac{2}{5}}\\ f_{n+\frac{3}{5}}\\ f_{n+\frac{4}{5}}\\ f_{n+1}
\end{bmatrix}
\qquad (4.5.3)
$$

Expanding (4.47) in Taylor series in the form

$$
\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{5}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{5} h y_n^{(1)} - \frac{1231}{126000} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{863}{50400} \left(\frac{1}{5}\right)^j \right. \\
\left. \qquad - \frac{761}{63000} \left(\frac{2}{5}\right)^j + \frac{941}{126000} \left(\frac{3}{5}\right)^j - \frac{341}{126000} \left(\frac{4}{5}\right)^j + \frac{107}{252000} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{5}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{2}{5} h y_n^{(1)} - \frac{71}{3150} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{544}{7875} \left(\frac{1}{5}\right)^j - \right. \\
\left. \qquad \frac{37}{1575} \left(\frac{2}{5}\right)^j + \frac{136}{7875} \left(\frac{3}{5}\right)^j - \frac{101}{15750} \left(\frac{4}{5}\right)^j + \frac{8}{7875} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{3}{5}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{3}{5} h y_n^{(1)} - \frac{123}{3500} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{3501}{28000} \left(\frac{1}{5}\right)^j \right. \\
\left. \qquad - \frac{9}{3500} \left(\frac{2}{5}\right)^j + \frac{87}{2800} \left(\frac{3}{5}\right)^j - \frac{9}{875} \left(\frac{4}{5}\right)^j + \frac{9}{5600} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{4}{5}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{4}{5} h y_n^{(1)} - \frac{376}{7875} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{1424}{7875} \left(\frac{1}{5}\right)^j \right. \\
\left. \qquad + \frac{176}{7875} \left(\frac{2}{5}\right)^j + \frac{608}{7875} \left(\frac{3}{5}\right)^j - \frac{16}{1575} \left(\frac{4}{5}\right)^j + \frac{16}{7875} \right] \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j)} - y_n - h y_n^{(1)} - \frac{61}{1008} h^2 y_n^2 - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{475}{2016} \left(\frac{1}{5}\right)^j \right. \\
\left. \qquad + \frac{25}{504} \left(\frac{2}{5}\right)^j + \frac{125}{1008} \left(\frac{3}{5}\right)^j + \frac{25}{1008} \left(\frac{4}{5}\right) + \frac{11}{2016} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{5}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{19}{288} h y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{1427}{7200} \left(\frac{1}{5}\right)^j - \frac{133}{1200} \left(\frac{2}{5}\right)^j \right. \\
\left. \qquad + \frac{241}{3600} \left(\frac{3}{5}\right)^j - \frac{173}{7200} \left(\frac{4}{5}\right)^j + \frac{3}{800} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{5}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{14}{225} h y_n^{(2)} \\
\qquad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{43}{150} \left(\frac{1}{5}\right)^j + \frac{7}{225} \left(\frac{2}{5}\right)^j + \frac{7}{225} \left(\frac{3}{5}\right)^j - \frac{1}{75} \left(\frac{4}{5}\right)^j + \frac{1}{450} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{3}{5}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{51}{800} h y_n^{(2)} \\
\qquad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{219}{800} \left(\frac{1}{5}\right)^j + \frac{57}{400} \left(\frac{2}{5}\right)^j + \frac{57}{400} \left(\frac{3}{5}\right)^j - \frac{21}{800} \left(\frac{4}{5}\right)^j + \frac{3}{800} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{4}{5}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{14}{225} h y_n^{(2)} \\
\qquad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{64}{225} \left(\frac{1}{5}\right)^j + \frac{8}{75} \left(\frac{2}{5}\right)^j + \frac{64}{226} \left(\frac{3}{5}\right)^j + \frac{14}{225} \left(\frac{4}{5}\right)^j \right] \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{19}{288} h y_n^2 \\
\qquad - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{25}{96} \left(\frac{1}{5}\right)^j + \frac{25}{144} \left(\frac{2}{5}\right)^j + \frac{25}{144} \left(\frac{3}{5}\right)^j + \frac{25}{96} \left(\frac{4}{5}\right)^j + \frac{19}{288} \right]
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad (4.5.4)
$$

and collecting terms in h and y leads to the following

$$\bar{c}_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{c}_1 = \begin{bmatrix} \frac{1}{5}-\frac{1}{5} \\ \frac{2}{5}-\frac{2}{5} \\ \frac{3}{5}-\frac{3}{5} \\ \frac{4}{5}-\frac{4}{5} \\ 1-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_2 = \begin{bmatrix} \left(\frac{1}{5}\right)^2\frac{1}{2!} - \frac{1231}{126000} - \left[\frac{863}{50400} - \frac{761}{63000} + \frac{941}{126000} - \frac{341}{126000} + \frac{107}{252000}\right] \\ \left(\frac{2}{5}\right)^2\frac{1}{2!} - \frac{71}{3150} - \left[\frac{544}{7875} - \frac{37}{1575} + \frac{136}{7875} - \frac{101}{15750} + \frac{8}{7875}\right] \\ \left(\frac{3}{5}\right)^2\frac{1}{2!} - \frac{123}{3500} - \left[\frac{3501}{28000} - \frac{9}{3500} + \frac{87}{2800} - \frac{9}{875} + \frac{9}{5600}\right] \\ \left(\frac{4}{5}\right)^2\frac{1}{2!} - \frac{376}{7875} - \left[\frac{1424}{7875} + \frac{176}{7875} + \frac{608}{7875} - \frac{16}{1575} + \frac{16}{7875}\right] \\ \frac{1}{2!} - \frac{61}{1008} - \left[\frac{475}{2016} + \frac{25}{504} + \frac{125}{1008} + \frac{25}{1008} + \frac{11}{2016}\right] \\ \frac{1}{5} - \frac{19}{288} - \left[\frac{1427}{7200} - \frac{133}{1200} + \frac{241}{3600} - \frac{173}{7200} + \frac{3}{800}\right] \\ \frac{2}{5} - \frac{14}{225} - \left[\frac{43}{150} + \frac{7}{225} + \frac{7}{225} - \frac{1}{75} + \frac{1}{450}\right] \\ \frac{3}{5} - \frac{51}{800} - \left[\frac{219}{800} + \frac{57}{400} + \frac{57}{400} - \frac{21}{800} + \frac{3}{800}\right] \\ \frac{4}{5} - \frac{14}{255} - \left[\frac{64}{225} + \frac{8}{75} + \frac{64}{225} + \frac{14}{225}\right] \\ 1 - \frac{19}{288} - \left[\frac{25}{96} + \frac{25}{144} + \frac{25}{144} + \frac{25}{96} + \frac{19}{288}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_3 = \begin{bmatrix} \left(\frac{1}{5}\right)^3 \frac{1}{3!} - \left[\frac{863}{50400}\left(\frac{1}{5}\right) - \frac{761}{63000}\left(\frac{2}{5}\right) + \frac{941}{126000}\left(\frac{3}{5}\right) - \frac{341}{12600}\left(\frac{4}{5}\right) + \frac{107}{25200}\right] \\ \left(\frac{2}{5}\right)^3 \frac{1}{3!} - \left[\frac{544}{7875}\left(\frac{1}{5}\right) - \frac{37}{1575}\left(\frac{2}{5}\right) + \frac{136}{7875}\left(\frac{3}{5}\right) - \frac{101}{15750}\left(\frac{4}{5}\right) + \frac{8}{7875}\right] \\ \left(\frac{3}{5}\right)^3 \frac{1}{3!} - \left[\frac{3501}{28000}\left(\frac{1}{5}\right) - \frac{9}{3500}\left(\frac{2}{5}\right) + \frac{87}{2800}\left(\frac{3}{5}\right) - \frac{9}{875}\left(\frac{4}{5}\right) + \frac{9}{5600}\right] \\ \left(\frac{4}{5}\right)^3 \frac{1}{3!} - \left[\frac{1424}{7875}\left(\frac{1}{5}\right) + \frac{176}{7875}\left(\frac{2}{5}\right) + \frac{608}{7875}\left(\frac{3}{5}\right) - \frac{16}{1575}\left(\frac{4}{5}\right) + \frac{16}{7875}\right] \\ \frac{1}{3!} - \left[\frac{475}{2016}\left(\frac{1}{5}\right) - \frac{25}{504}\left(\frac{2}{5}\right) + \frac{125}{1008}\left(\frac{3}{5}\right) - \frac{25}{1008}\left(\frac{4}{5}\right) + \frac{11}{2016}\right] \\ \left(\frac{1}{5}\right)^2 \frac{1}{2!} - \left[\frac{1427}{7200}\left(\frac{1}{5}\right) - \frac{133}{1200}\left(\frac{2}{5}\right) + \frac{241}{3600}\left(\frac{3}{5}\right) - \frac{173}{7200}\left(\frac{4}{5}\right) + \frac{3}{800}\right] \\ \left(\frac{2}{5}\right)^2 \frac{1}{2!} - \left[\frac{43}{150}\left(\frac{1}{5}\right) + \frac{7}{225}\left(\frac{2}{5}\right) + \frac{7}{225}\left(\frac{3}{5}\right) - \frac{1}{75}\left(\frac{4}{5}\right) + \frac{1}{450}\right] \\ \left(\frac{3}{5}\right)^2 \frac{1}{2!} - \left[\frac{219}{800}\left(\frac{1}{5}\right) + \frac{57}{400}\left(\frac{2}{5}\right) + \frac{57}{400}\left(\frac{3}{5}\right) - \frac{21}{800}\left(\frac{4}{5}\right) + \frac{3}{800}\right] \\ \left(\frac{4}{5}\right)^2 \frac{1}{2!} - \left[\frac{64}{225}\left(\frac{1}{5}\right) + \frac{8}{75}\left(\frac{2}{5}\right) + \frac{64}{225}\left(\frac{3}{5}\right) + \frac{14}{225}\left(\frac{4}{5}\right)\right] \\ \frac{1}{2!} - \left[\frac{25}{96}\left(\frac{1}{5}\right) + \frac{25}{144}\left(\frac{2}{5}\right) + \frac{25}{144}\left(\frac{3}{5}\right) + \frac{25}{96}\left(\frac{4}{5}\right) + \frac{19}{288}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_4 = \begin{bmatrix} \left(\frac{1}{5}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{761}{50400}\left(\frac{1}{5}\right)^2 - \frac{761}{63000}\left(\frac{2}{5}\right)^2 + \frac{941}{126000}\left(\frac{3}{5}\right)^2 - \frac{341}{126000}\left(\frac{4}{5}\right)^2 + \frac{107}{252000}\right] \\ \left(\frac{2}{5}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{544}{7875}\left(\frac{1}{5}\right)^2 - \frac{37}{1575}\left(\frac{2}{5}\right)^2 + \frac{136}{7875}\left(\frac{3}{5}\right)^2 - \frac{101}{15750}\left(\frac{4}{5}\right)^2 + \frac{8}{7875}\right] \\ \left(\frac{3}{5}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{3501}{28000}\left(\frac{1}{5}\right)^2 - \frac{9}{3500}\left(\frac{2}{5}\right)^2 - \frac{87}{2800}\left(\frac{3}{5}\right)^2 - \frac{9}{875}\left(\frac{4}{5}\right)^2 + \frac{9}{5600}\right] \\ \left(\frac{4}{5}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{1424}{7875}\left(\frac{1}{5}\right)^2 + \frac{176}{7875}\left(\frac{2}{5}\right)^2 + \frac{608}{7875}\left(\frac{3}{5}\right)^2 - \frac{16}{1575}\left(\frac{4}{5}\right)^2 + \frac{16}{7875}\right] \\ \frac{1}{4!} - \frac{1}{2!}\left[\frac{475}{2016}\left(\frac{1}{5}\right)^2 + \frac{25}{504}\left(\frac{2}{5}\right)^2 + \frac{125}{1008}\left(\frac{3}{5}\right)^2 + \frac{25}{1008}\left(\frac{4}{5}\right)^2 + \frac{11}{2016}\right] \\ \left(\frac{1}{5}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{1427}{7200}\left(\frac{1}{5}\right)^2 - \frac{133}{1200}\left(\frac{2}{5}\right)^2 + \frac{241}{3600}\left(\frac{3}{5}\right)^2 - \frac{173}{7200}\left(\frac{4}{5}\right)^2 + \frac{3}{800}\right] \\ \left(\frac{2}{5}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{43}{150}\left(\frac{1}{5}\right)^2 + \frac{7}{225}\left(\frac{2}{5}\right)^2 + \frac{7}{225}\left(\frac{3}{5}\right)^2 - \frac{1}{75}\left(\frac{4}{5}\right)^2 + \frac{1}{450}\right] \\ \left(\frac{3}{5}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{219}{800}\left(\frac{1}{5}\right)^2 + \frac{57}{400}\left(\frac{2}{5}\right)^2 + \frac{57}{400}\left(\frac{3}{5}\right)^2 - \frac{21}{800}\left(\frac{4}{5}\right)^2 + \frac{3}{800}\right] \\ \left(\frac{4}{5}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{64}{225}\left(\frac{1}{5}\right)^2 + \frac{8}{75}\left(\frac{2}{5}\right)^2 + \frac{64}{225}\left(\frac{3}{5}\right)^2 + \frac{14}{225}\left(\frac{4}{5}\right)^2\right] \\ \frac{1}{3!} - \frac{1}{2!}\left[\frac{25}{96}\left(\frac{1}{5}\right)^2 + \frac{25}{144}\left(\frac{2}{5}\right)^2 + \frac{25}{144}\left(\frac{3}{5}\right)^2 + \frac{25}{96}\left(\frac{4}{5}\right)^2 + \frac{19}{288}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_5 = \begin{bmatrix} \left(\frac{1}{5}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{863}{50400}\left(\frac{1}{5}\right)^3 - \frac{761}{63000}\left(\frac{2}{5}\right)^3 + \frac{941}{126000}\left(\frac{3}{5}\right)^3 - \frac{341}{126000}\left(\frac{4}{5}\right)^3 + \frac{107}{252000}\right] \\ \left(\frac{2}{5}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{544}{7875}\left(\frac{1}{5}\right)^3 - \frac{37}{1575}\left(\frac{2}{5}\right)^3 + \frac{136}{7875}\left(\frac{3}{5}\right)^3 - \frac{101}{15750}\left(\frac{4}{5}\right)^3 + \frac{8}{7875}\right] \\ \left(\frac{3}{5}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{3501}{28000}\left(\frac{1}{5}\right)^3 - \frac{9}{3500}\left(\frac{2}{5}\right)^3 + \frac{87}{2800}\left(\frac{3}{5}\right)^3 - \frac{9}{875}\left(\frac{4}{5}\right)^3 + \frac{9}{5600}\right] \\ \left(\frac{4}{5}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{1424}{7879}\left(\frac{1}{5}\right)^3 + \frac{176}{7875}\left(\frac{2}{5}\right)^3 + \frac{608}{7875}\left(\frac{3}{5}\right)^3 - \frac{16}{1575}\left(\frac{4}{5}\right)^3 + \frac{16}{7875}\right] \\ \frac{1}{5!} - \frac{1}{3!}\left[\frac{475}{2016}\left(\frac{1}{5}\right)^3 + \frac{25}{504}\left(\frac{2}{5}\right)^3 + \frac{125}{1008}\left(\frac{3}{5}\right)^3 + \frac{25}{1008}\left(\frac{4}{5}\right)^3 + \frac{11}{2016}\right] \\ \left(\frac{1}{5}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{1427}{7200}\left(\frac{1}{5}\right)^3 - \frac{133}{1200}\left(\frac{2}{5}\right)^3 + \frac{241}{3600}\left(\frac{3}{5}\right)^3 - \frac{173}{7200}\left(\frac{4}{5}\right)^3 + \frac{3}{800}\right] \\ \left(\frac{2}{5}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{43}{150}\left(\frac{1}{5}\right)^3 + \frac{7}{225}\left(\frac{2}{5}\right)^3 + \frac{7}{225}\left(\frac{3}{5}\right)^3 - \frac{1}{75}\left(\frac{4}{5}\right)^3 + \frac{1}{450}\right] \\ \left(\frac{3}{5}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{219}{800}\left(\frac{1}{5}\right)^3 + \frac{57}{400}\left(\frac{2}{5}\right)^3 + \frac{57}{400}\left(\frac{3}{5}\right)^3 - \frac{21}{800}\left(\frac{4}{5}\right)^3 + \frac{3}{800}\right] \\ \left(\frac{4}{5}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{64}{225}\left(\frac{1}{5}\right)^3 + \frac{8}{75}\left(\frac{2}{5}\right)^3 + \frac{64}{225}\left(\frac{3}{5}\right)^3 + \frac{14}{225}\left(\frac{4}{5}\right)^3\right] \\ \frac{1}{4!} - \frac{1}{3!}\left[\frac{25}{96}\left(\frac{1}{5}\right)^3 + \frac{25}{144}\left(\frac{2}{5}\right)^3 + \frac{25}{144}\left(\frac{3}{5}\right)^3 + \frac{25}{96}\left(\frac{4}{5}\right)^3 + \frac{19}{288}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{c}_6 = \begin{bmatrix} \left(\frac{1}{5}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{863}{50400}\left(\frac{1}{5}\right)^4 - \frac{761}{63000}\left(\frac{2}{5}\right)^4 + \frac{941}{126000}\left(\frac{3}{5}\right)^4 - \frac{341}{126000}\left(\frac{4}{5}\right)^4 + \frac{107}{252000}\right] \\ \left(\frac{2}{5}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{544}{7875}\left(\frac{1}{5}\right)^4 - \frac{37}{1575}\left(\frac{2}{5}\right)^4 + \frac{136}{7875}\left(\frac{3}{5}\right)^4 - \frac{101}{15750}\left(\frac{4}{5}\right)^4 + \frac{8}{7875}\right] \\ \left(\frac{3}{5}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{3501}{28000}\left(\frac{1}{5}\right)^4 - \frac{9}{3500}\left(\frac{2}{5}\right)^4 + \frac{87}{2800}\left(\frac{3}{5}\right)^4 - \frac{9}{875}\left(\frac{4}{5}\right)^4 + \frac{9}{5600}\right] \\ \left(\frac{4}{5}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{1424}{7879}\left(\frac{1}{5}\right)^4 + \frac{176}{7875}\left(\frac{2}{5}\right)^4 + \frac{608}{7875}\left(\frac{3}{5}\right)^4 - \frac{16}{1575}\left(\frac{4}{5}\right)^4 + \frac{16}{7875}\right] \\ \frac{1}{6!} - \frac{1}{4!}\left[\frac{475}{2016}\left(\frac{1}{5}\right)^4 + \frac{25}{504}\left(\frac{2}{5}\right)^4 + \frac{125}{1008}\left(\frac{3}{5}\right)^4 + \frac{25}{1008}\left(\frac{4}{5}\right)^4 + \frac{11}{2016}\right] \\ \left(\frac{1}{5}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{1427}{7200}\left(\frac{1}{5}\right)^4 - \frac{133}{1200}\left(\frac{2}{5}\right)^4 + \frac{241}{3600}\left(\frac{3}{5}\right)^4 - \frac{173}{7200}\left(\frac{4}{5}\right)^4 + \frac{3}{800}\right] \\ \left(\frac{2}{5}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{43}{150}\left(\frac{1}{5}\right)^4 + \frac{7}{225}\left(\frac{2}{5}\right)^4 + \frac{7}{225}\left(\frac{3}{5}\right)^4 - \frac{1}{75}\left(\frac{4}{5}\right)^4 + \frac{1}{450}\right] \\ \left(\frac{3}{5}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{219}{800}\left(\frac{1}{5}\right)^4 + \frac{57}{400}\left(\frac{2}{5}\right)^4 + \frac{57}{400}\left(\frac{3}{5}\right)^4 - \frac{21}{800}\left(\frac{4}{5}\right)^4 + \frac{3}{800}\right] \\ \left(\frac{4}{5}\right)^5 \frac{1}{5!} - \frac{1}{4!}\left[\frac{64}{225}\left(\frac{1}{5}\right)^4 + \frac{8}{75}\left(\frac{2}{5}\right)^4 + \frac{64}{225}\left(\frac{3}{5}\right)^4 + \frac{14}{225}\left(\frac{4}{5}\right)^4\right] \\ \frac{1}{5!} - \frac{1}{4!}\left[\frac{25}{96}\left(\frac{1}{5}\right)^4 + \frac{25}{144}\left(\frac{2}{5}\right)^4 + \frac{25}{144}\left(\frac{3}{5}\right)^4 + \frac{25}{96}\left(\frac{4}{5}\right)^4 + \frac{19}{288}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{c}_7 \quad = \quad \begin{bmatrix}
\left(\frac{1}{5}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{863}{50400}\left(\frac{1}{5}\right)^5 - \frac{761}{63000}\left(\frac{2}{5}\right)^5 + \frac{941}{126000}\left(\frac{3}{5}\right)^5 - \frac{341}{126000}\left(\frac{4}{5}\right)^5 + \frac{107}{252000}\right] \\
\left(\frac{2}{5}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{544}{7875}\left(\frac{1}{5}\right)^5 - \frac{37}{1575}\left(\frac{2}{5}\right)^5 + \frac{136}{7875}\left(\frac{3}{5}\right)^5 - \frac{101}{15750}\left(\frac{4}{5}\right)^5 + \frac{8}{7875}\right] \\
\left(\frac{3}{5}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{3501}{28000}\left(\frac{1}{5}\right)^5 - \frac{9}{3500}\left(\frac{2}{5}\right)^5 + \frac{87}{2800}\left(\frac{3}{5}\right)^5 - \frac{9}{875}\left(\frac{4}{5}\right)^5 + \frac{9}{5600}\right] \\
\left(\frac{4}{5}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{1424}{7879}\left(\frac{1}{5}\right)^5 + \frac{176}{7875}\left(\frac{2}{5}\right)^5 + \frac{608}{7875}\left(\frac{3}{5}\right)^5 - \frac{16}{1575}\left(\frac{4}{5}\right)^5 + \frac{16}{7875}\right] \\
\frac{1}{7!} - \frac{1}{5!}\left[\frac{475}{2016}\left(\frac{1}{5}\right)^5 + \frac{25}{504}\left(\frac{2}{5}\right)^5 + \frac{125}{1008}\left(\frac{3}{5}\right)^5 + \frac{25}{1008}\left(\frac{4}{5}\right)^5 + \frac{11}{2016}\right] \\
\left(\frac{1}{5}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{1427}{7200}\left(\frac{1}{5}\right)^5 - \frac{133}{1200}\left(\frac{2}{5}\right)^5 + \frac{241}{3600}\left(\frac{3}{5}\right)^5 - \frac{173}{7200}\left(\frac{4}{5}\right)^5 + \frac{3}{800}\right] \\
\left(\frac{2}{5}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{43}{150}\left(\frac{1}{5}\right)^5 + \frac{7}{225}\left(\frac{2}{5}\right)^5 + \frac{7}{225}\left(\frac{3}{5}\right)^5 - \frac{1}{75}\left(\frac{4}{5}\right)^5 + \frac{1}{450}\right] \\
\left(\frac{3}{5}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{219}{800}\left(\frac{1}{5}\right)^5 + \frac{57}{400}\left(\frac{2}{5}\right)^5 + \frac{57}{400}\left(\frac{3}{5}\right)^5 - \frac{21}{800}\left(\frac{4}{5}\right)^5 + \frac{3}{800}\right] \\
\left(\frac{4}{5}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{64}{225}\left(\frac{1}{5}\right)^5 + \frac{8}{75}\left(\frac{2}{5}\right)^5 + \frac{64}{225}\left(\frac{3}{5}\right)^5 + \frac{14}{225}\left(\frac{4}{5}\right)^5\right] \\
\frac{1}{6!} - \frac{1}{5!}\left[\frac{25}{96}\left(\frac{1}{5}\right)^5 + \frac{25}{144}\left(\frac{2}{5}\right)^5 + \frac{25}{144}\left(\frac{3}{5}\right)^5 + \frac{25}{96}\left(\frac{4}{5}\right)^5 + \frac{19}{288}\right]
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$\bar{c}_8 = \begin{bmatrix} \left(\frac{1}{5}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{863}{50400}\left(\frac{1}{5}\right)^6 - \frac{761}{63000}\left(\frac{2}{5}\right)^6 + \frac{941}{126000}\left(\frac{3}{5}\right)^6 - \frac{341}{126000}\left(\frac{4}{5}\right)^6 + \frac{107}{252000}\right] \\ \left(\frac{2}{5}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{544}{7875}\left(\frac{1}{5}\right)^6 - \frac{37}{1575}\left(\frac{2}{5}\right)^6 + \frac{136}{7875}\left(\frac{3}{5}\right)^6 - \frac{101}{15750}\left(\frac{4}{5}\right)^6 + \frac{8}{7875}\right] \\ \left(\frac{3}{5}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{3501}{28000}\left(\frac{1}{5}\right)^6 - \frac{9}{3500}\left(\frac{2}{5}\right)^6 + \frac{87}{2800}\left(\frac{3}{5}\right)^6 - \frac{9}{875}\left(\frac{4}{5}\right)^6 + \frac{9}{5600}\right] \\ \left(\frac{4}{5}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{1424}{7875}\left(\frac{1}{5}\right)^6 + \frac{176}{7875}\left(\frac{2}{5}\right)^6 + \frac{608}{7875}\left(\frac{3}{5}\right)^6 - \frac{16}{1575}\left(\frac{4}{5}\right)^6 + \frac{16}{7875}\right] \\ \frac{1}{8!} - \frac{1}{6!}\left[\frac{475}{2016}\left(\frac{1}{5}\right)^6 + \frac{25}{504}\left(\frac{2}{5}\right)^6 + \frac{125}{1008}\left(\frac{3}{5}\right)^6 + \frac{25}{1008}\left(\frac{4}{5}\right)^6 + \frac{11}{2016}\right] \\ \left(\frac{1}{5}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{1427}{7200}\left(\frac{1}{5}\right)^6 - \frac{133}{1200}\left(\frac{2}{5}\right)^6 + \frac{241}{3600}\left(\frac{3}{5}\right)^6 - \frac{173}{7200}\left(\frac{4}{5}\right)^6 + \frac{3}{800}\right] \\ \left(\frac{2}{5}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{43}{150}\left(\frac{1}{5}\right)^6 + \frac{7}{225}\left(\frac{2}{5}\right)^6 + \frac{7}{225}\left(\frac{3}{5}\right)^6 - \frac{1}{75}\left(\frac{4}{5}\right)^6 + \frac{1}{450}\right] \\ \left(\frac{3}{5}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{219}{800}\left(\frac{1}{5}\right)^6 + \frac{57}{400}\left(\frac{2}{5}\right)^6 + \frac{57}{400}\left(\frac{3}{5}\right)^6 - \frac{21}{800}\left(\frac{4}{5}\right)^6 + \frac{3}{800}\right] \\ \left(\frac{4}{5}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{64}{225}\left(\frac{1}{5}\right)^6 + \frac{8}{75}\left(\frac{2}{5}\right)^6 + \frac{64}{225}\left(\frac{3}{5}\right)^6 + \frac{14}{225}\left(\frac{4}{5}\right)^6\right] \\ \frac{1}{7!} - \frac{1}{6!}\left[\frac{25}{96}\left(\frac{1}{5}\right)^6 + \frac{25}{144}\left(\frac{2}{5}\right)^6 + \frac{25}{144}\left(\frac{3}{5}\right)^6 + \frac{25}{96}\left(\frac{4}{5}\right)^6 + \frac{19}{288}\right] \end{bmatrix} = \begin{bmatrix} -\frac{199}{9450000000} \\ -\frac{19}{369140625} \\ -\frac{141}{1750000000} \\ -\frac{8}{73828125} \\ -\frac{11}{75600000} \\ -\frac{863}{4725000000} \\ -\frac{37}{295312500} \\ -\frac{29}{175000000} \\ -\frac{8}{73828125} \\ -\frac{11}{37800000} \end{bmatrix}$$

Hence the block method (3.3.51) is of order $p = (6,6,6,6,6,6,6,6,6,6)^T$ with error constant $c_{p+2} = \left(-\frac{199}{4950000000}, -\frac{19}{369140625}, -\frac{141}{1750000000}, -\frac{8}{73828125}, -\frac{11}{75600000}, -\frac{863}{4725000000}, -\frac{37}{295312500}, -\frac{29}{175000000}, -\frac{8}{73828125}, -\frac{11}{37800000}\right)^T$.

4.5.2 Consistency

The block method (3.3.51) has order $p = (6,6,6,6,6,6,6,6,6,6)^T > 1$, therefore, by condition (i) in definition (4.1.6) it is consistent. Following the consistency of the block method (3.3.51), the consistency of method (3.3.47d) is shown as follows by conditions (i) - (iv) in definition (4.1.6).

Condition (i)

The main method (3.3.47d) is of order $p = 6 > 1$, which satisfies condition (i) of definition (4.1.6).

Condition (ii)

The first and second characteristic polynomials of method (3.3.47d) are given respectively by

$$\rho(z) = z - 2z^{4/5} + z^{3.5} \tag{4.5.5}$$

and

$$\sigma(z) = \frac{18z + 209z^{4/5} + 4z^{3/5} + 14z^{2/5} - 6z^{1/5} + 1}{6000} \tag{4.5.6}$$

From (4.5.5), $\alpha_{3/5} = 1$, $\alpha_{4/5} = -2$ and $\alpha_1 = 1$ therefore

$$\sum_j \alpha_j = 1 - 2 + 1 = 0, \quad j = \frac{3}{5},\ \frac{4}{5},\ 1$$

Condition (iii)

From (4.5.5),

$$\begin{aligned} \rho'(z) &= \frac{1}{5}z^{-2/5}(5z^{2/5} - 8z^{1/5} + 3) \\ \Rightarrow \quad \rho'(1) &= \tfrac{1}{5}(5 - 8 + 3) = 0 \end{aligned} \tag{4.5.7}$$

Also by (4.5.5)

$$\begin{aligned} \rho(1) &= 1 - 2 + 1 = 0 \\ \rho(1) &= \rho'(1) = 0 \end{aligned}$$

Condition (iv)

From (4.5.7),

$$\begin{aligned} \rho''(z) &= \frac{2}{25}z^{-7/5}(4z^{1/5} - 3) \\ &= \frac{2}{25}(4 - 3) = \frac{2}{25} \end{aligned}$$

From (4.5.6),

$$\begin{aligned} \sigma(1) &= \frac{18 + 209 + 4 + 14 - 6 + 1}{6000} = \frac{1}{25} \\ \Rightarrow 2!\sigma(1) &= 2\left(\frac{1}{25}\right) = \frac{2}{25} \\ \therefore\ \rho''(1) &= 2!\sigma(1) \end{aligned}$$

Thus by definition (4.1.6), the main method (3.3.47d) is consistent.

4.5.3 Zero Stability of the One Step Method with Four Off-step Points

4.5.3.1 Zero stability of the Block Method (3.3.51)

Using (3.3.51) and (3.2.12) in the limit as $h \to 0$ in (4.1.9), the first characteristic polynomial of the block method is obtained (by equation (4.1.9)) as

$$\rho(z) = \det \left[z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right]$$

i.e.

$$\rho(r) = \det \begin{bmatrix} z & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{bmatrix} = z^{10} - z^9 \tag{4.5.8}$$

Equating (4.5.8) to zero and solving for z gives $z = 0$ or 1.

Since no root has modulus greater than one and $|z| = 1$ is simple, the block method is zero stable in the limit as $h \to 0$ by definition (4.1.8).

4.5.3.2 Zero Stability of the Main Method (3.3.47d)

Equation (4.5.5) is the first characteristic polynomial of the main method (3.3.47d). Equating (4.5.5) to zero and solving for z gives $z = 0$ or $z = 1$.

We can see that no root has modulus greater than one and $|z| = 1$ is simple. It follows from definition (4.1.7) that the method is zero stable.

4.5.4 Convergence

Convergence of the main method (3.3.47d) follows from Theorem 4.1.1.

4.5.5 Region of absolute stability of the One step method with four offstep points

The first and second characteristics polynomials of the main method (3.3.47d) were given as (4.5.5) and (4.5.6) respectively. Hence, by the boundary locus method, the boundary of the region of absolute stability is given by (4.1.14) where $z = e^{i\theta}$ as

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} = \frac{6000(e^{i\theta} - 2e^{i\frac{4}{5}\theta} + e^{i\frac{3}{5}\theta})}{18e^{i\theta} + 209e^{i\frac{4}{5}\theta} + 4e^{i\frac{3}{5}\theta} + 14e^{i\frac{2}{5}\theta} - 6e^{i\frac{1}{5}\theta} + 1} \tag{4.5.9}$$

where $e^{i\theta} = z$ is the value of the root of the stability polynomial (4.1.10). Recall that $e^{i\theta} = \cos\theta + i\sin\theta$, thus (4.53) reduces to

$$\bar{h}(\theta) = \frac{2328000\cos\frac{1}{5}\theta - 72000\cos\frac{2}{5}\theta + 162000\cos\frac{3}{5}\theta - 48000\cos\frac{4}{5}\theta + 6000\cos\theta - 2376000}{9128\cos\frac{1}{5}\theta + 5976\cos\frac{2}{5}\theta - 1996\cos\frac{3}{5}\theta + 202\cos\frac{4}{5}\theta + 36\cos\theta + 44254} \tag{4.5.10}$$

Evaluating (4.5.10) at intervals of 30^0 gives the results in table 4.

Table 4. The boundaries of the region of absolute stability of the one step 4 offstep points method.

θ	0	30^0	60^0	90^0	120^0	150^0	180^0
$h(\theta)$	0	-0.27	-1.10	-2.47	-6.04	-6.85	-9.87

From table 4, the interval of absolute stability is $(-9.87, 0)$. The region of absolute stability is shown in figure 4.4.

Remark

The locus is symmetric about x-axis that is $x(-\theta) = x(\theta)$. Similarly, $y(-\theta) = y(\theta)$.

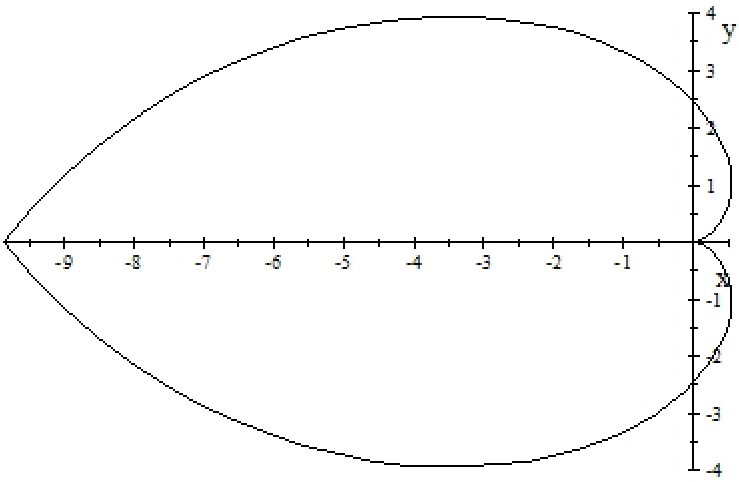

Figure 4.4: Region of Absolute Stability of the Continuous One Step Method with Four Offstep Points

4.6 Analysis of the One Step Method with Five Offstep Points

The basic properties of the main method (3.3.60e) such as order, error constant, consistency and zero stability as well as those of the associated block method (3.3.64) are obtained in this section. The region of absolute stability of the main method (3.3.60e) is also obtained here.

4.6.1 Order and Error Constant

4.6.1.1 Order and error constant of the main method (3.3.60e)

Let (3.3.60e) be written in the form

$$y_{n+1} - 2y_{n+\frac{5}{6}} - y_{n+\frac{2}{3}} - \frac{h}{2177280}\Big[4315f_{n+1} + 53994f_{n+\frac{5}{6}} - 2307f_{n+\frac{2}{3}} + 7948f_{n+\frac{1}{2}} - 4827f_{n+\frac{1}{3}} + 1578f_{n+\frac{1}{6}} - 221f_n\Big] \quad (4.5.11)$$

Expanding in Taylor series in the form

$$\sum_{j=1}^{\infty}\frac{h^j}{j!}y_n^{(j)}-2\sum_{j=0}^{\infty}\frac{\left(\frac{5}{6}\right)^j h^j}{j!}y_n^{(j)}+\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j h^j}{j!}y_n^{(j)}+\frac{221}{2177280}h^2y_n^{(1)}-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left(\frac{1}{6}\right)^j$$
$$\left[\frac{263}{362880}-\frac{1609}{725760}\left(\frac{1}{3}\right)^j+\frac{1987}{544320}\left(\frac{1}{2}\right)^j-\frac{769}{725760}\left(\frac{2}{3}\right)^j+\frac{8999}{362880}\left(\frac{5}{6}\right)^j+\frac{863}{435456}\right] \quad (4.5.12)$$

and collecting terms in powers of h and y leads to the following:

$$\begin{aligned}
c_0 &= 1-2+1=0\\
c_1 &= 1-2\left(\frac{5}{6}\right)+\frac{2}{3}=0\\
c_2 &= \frac{1}{2!}-2\left(\frac{5}{6}\right)^2\frac{1}{2!}+\left(\frac{2}{3}\right)^2\frac{1}{2!}+\frac{221}{2177280}-\left[\frac{263}{362880}-\frac{1609}{725760}+\frac{1987}{544320}-\frac{769}{725760}\right.\\
&\qquad\left.+\frac{8999}{362880}+\frac{863}{435456}\right]=0\\
c_3 &= \frac{1}{3!}-2\left(\frac{5}{6}\right)^3\frac{1}{3!}+\left(\frac{2}{3}\right)^3\frac{1}{3!}-\left[\frac{263}{362880}\left(\frac{1}{6}\right)-\frac{1609}{725760}\left(\frac{1}{3}\right)+\frac{1987}{544320}\left(\frac{1}{2}\right)\right.\\
&\qquad\left.-\frac{769}{725760}\left(\frac{2}{3}\right)+\frac{8999}{362880}\left(\frac{5}{6}\right)+\frac{863}{435456}\right]=0\\
c_4 &= \frac{1}{4!}-2\left(\frac{5}{6}\right)^4\frac{1}{4!}+\left(\frac{2}{3}\right)^4\frac{1}{4!}-\frac{1}{2!}\left[\frac{263}{362880}\left(\frac{1}{6}\right)^2-\frac{1609}{725760}\left(\frac{1}{3}\right)^2+\frac{1987}{544320}\left(\frac{1}{2}\right)^2\right.\\
&\qquad\left.-\frac{769}{725760}\left(\frac{2}{3}\right)^2+\frac{8999}{362880}\left(\frac{5}{6}\right)+\frac{863}{435456}\right]=0
\end{aligned}$$

$$
\begin{aligned}
c_5 &= \frac{1}{5!} - 2\left(\frac{5}{6}\right)^5 \frac{1}{5!} + \left(\frac{2}{5}\right)^5 \frac{1}{5!} - \frac{1}{3}\left[\frac{263}{362880}\left(\frac{1}{6}\right)^2 - \frac{1609}{725760}\left(\frac{1}{3}\right)^3 + \frac{1987}{544320}\left(\frac{1}{2}\right)^3\right. \\
&\quad \left. - \frac{769}{725760}\left(\frac{2}{3}\right)^3 + \frac{8999}{362880}\left(\frac{5}{6}\right)^3 + \frac{863}{435456}\right] = 0 \\
c_6 &= \frac{1}{6!} - 2\left(\frac{5}{6}\right)^6 \frac{1}{6!} + \left(\frac{2}{3}\right)^6 \frac{1}{6!} - \frac{1}{4!}\left[\frac{263}{362880}\left(\frac{1}{6}\right)^4 - \frac{1609}{725760}\left(\frac{1}{3}\right)^4 + \frac{1987}{544320}\left(\frac{1}{2}\right)^4\right. \\
&\quad \left. - \frac{769}{725760}\left(\frac{2}{3}\right)^4 + \frac{8999}{362880}^4 + \frac{863}{435456}\right] = 0 \\
c_7 &= \frac{1}{7!} - 2\left(\frac{5}{6}\right)^7 \frac{1}{7!} + \left(\frac{2}{3}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{263}{362880}\left(\frac{1}{6}\right)^5 - \frac{1609}{725760}\left(\frac{1}{3}\right)^5 + \frac{1987}{544320}\left(\frac{1}{2}\right)^5\right. \\
&\quad \left. - \frac{769}{725760}\left(\frac{2}{3}\right)^5 + \frac{8999}{362880}\left(\frac{5}{6}\right)^5 + \frac{863}{435456}\right] = 0 \\
c_8 &= \frac{1}{8!} - 2\left(\frac{5}{6}\right)^8 \frac{1}{8!} + \left(\frac{2}{3}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{263}{362880}\left(\frac{1}{6}\right)^6 - \frac{1609}{725760}\left(\frac{1}{3}\right)^6 + \frac{1987}{544320}\left(\frac{1}{2}\right)^6\right. \\
&\quad \left. - \frac{769}{725760}\left(\frac{2}{3}\right)^6 + \frac{8999}{362880}\left(\frac{5}{6}\right)^6 + \frac{863}{435456}\right] = 0
\end{aligned}
$$

$$
\begin{aligned}
c_9 &= \frac{1}{9!} - 2\left(\frac{5}{6}\right)^9 \frac{1}{9!} + \left(\frac{2}{3}\right)^9 \frac{1}{9!} - \frac{1}{7!}\left[\frac{263}{362880}\left(\frac{1}{6}\right)^7 - \frac{1609}{725760}\left(\frac{1}{3}\right)^7 + \frac{1987}{544320}\left(\frac{1}{2}\right)^7\right. \\
&\quad \left. - \frac{769}{725760}\left(\frac{2}{3}\right)^7 + \frac{8999}{362880}\left(\frac{5}{6}\right)^7 + \frac{863}{435456}\right] = \frac{-19}{60949905408}
\end{aligned}
$$

Hence, the main method (3.3.60e) has order $p = 7$ with error constant $c_{p+2} = -3.1173 \times 10^{-10}$.

4.6.1.2 Order of the Block Method (3.3.64)

Let (3.3.64) be expressed in the form

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{2}{3}}\\ y_{n+\frac{5}{6}}\\ y_{n+1}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{3}}\\ y'_{n+\frac{1}{2}}\\ y'_{n+\frac{2}{3}}\\ y'_{n+\frac{5}{6}}\\ y'_{n+1} \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{6}h\\ 1 & \frac{1}{3}h\\ 1 & \frac{1}{2}h\\ 1 & \frac{2}{3}h\\ 1 & \frac{5}{6}h\\ 1 & h\\ 0&1\\ 0&1\\ 0&1\\ 0&1\\ 0&1\\ 0&1 \end{bmatrix} \begin{bmatrix} y_n\\ y'_n \end{bmatrix} - \begin{bmatrix} \frac{28549}{4354560}h^2\\ \frac{1027}{68040}h^2\\ \frac{253}{10752}h^2\\ \frac{272}{8505}h^2\\ \frac{35225}{870912}h^2\\ \frac{41}{840}h^2\\ \frac{19087}{362880}h\\ \frac{1139}{22680}h\\ \frac{137}{2688}\\ \frac{143}{2835}h\\ \frac{3715}{72576}h\\ \frac{41}{840}h \end{bmatrix} [f_n]$$

$$
-\begin{bmatrix}
\frac{275}{20736}h^2 & -\frac{5717}{483840}h^2 & \frac{10621}{1088640}h^2 & -\frac{7703}{1451520}h^2 & \frac{403}{241920}h^2 & -\frac{199}{870912}h^2 \\
\frac{97}{1890}h^2 & -\frac{2}{81}h^2 & \frac{197}{8505}h^2 & -\frac{97}{7560}h^2 & \frac{23}{5670}h^2 & -\frac{19}{34020}h^2 \\
\frac{165}{1792}h^2 & -\frac{267}{17920}h^2 & \frac{5}{128}h^2 & -\frac{363}{17920}h^2 & \frac{57}{8960}h^2 & -\frac{47}{53760}h^2 \\
\frac{376}{2835}h^2 & -\frac{2}{945}h^2 & \frac{656}{8505}h^2 & -\frac{2}{81}h^2 & \frac{8}{945}h^2 & -\frac{2}{1701}h^2 \\
\frac{8375}{48384}h^2 & \frac{3125}{290304}h^2 & \frac{25625}{217728}h^2 & -\frac{625}{96768}h^2 & \frac{275}{20736}h^2 & -\frac{1375}{870912}h^2 \\
\frac{3}{14}h^2 & \frac{3}{140}h^2 & \frac{17}{105}h^2 & \frac{3}{280}h^2 & \frac{3}{70}h^2 & 0 \\
\frac{2713}{15120}h & -\frac{15487}{120960}h & \frac{293}{2835}h & -\frac{6737}{120960}h & \frac{263}{15120}h & -\frac{863}{362880}h \\
\frac{47}{189}h & \frac{11}{7560}h & \frac{166}{2835}h & -\frac{269}{7560}h & \frac{11}{945}h & -\frac{37}{22680}h \\
\frac{27}{112}h & \frac{387}{4480}h & \frac{17}{105}h & -\frac{243}{4480}h & \frac{9}{560}h & -\frac{29}{13440}h \\
\frac{232}{945}h & \frac{64}{945}h & \frac{752}{2835}h & \frac{29}{945}h & \frac{8}{945}h & -\frac{4}{2835}h \\
\frac{725}{3024}h & \frac{2125}{24192}h & \frac{125}{567}h & \frac{3875}{24192}h & \frac{235}{3024}h & -\frac{275}{72576}h \\
\frac{9}{35}h & \frac{9}{280}h & \frac{34}{105}h & \frac{9}{280}h & \frac{9}{35}h & \frac{41}{840}h
\end{bmatrix}
\begin{bmatrix}
f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{5}{6}} \\ f_{n+1}
\end{bmatrix}
\tag{4.5.13}
$$

Expanding (4.5.13) in Taylor series in the form

$$\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{6}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{6} h y_n^{(1)} - \frac{28549}{4354560} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{275}{20736}\left(\frac{1}{6}\right)^j - \frac{5717}{483840}\left(\frac{1}{3}\right)^j + \frac{10621}{1088640}\left(\frac{1}{2}\right)^j - \frac{7703}{1451520}\left(\frac{2}{3}\right)^j + \frac{403}{241920}\left(\frac{5}{6}\right) - \frac{199}{870912} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{3} h y_n^{(1)} - \frac{1027}{68040} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{97}{1890}\left(\frac{1}{6}\right)^j - \frac{2}{81}\left(\frac{1}{3}\right)^j + \frac{197}{8505}\left(\frac{1}{2}\right)^j - \frac{97}{7560}\left(\frac{2}{3}\right)^j + \frac{23}{5670}\left(\frac{5}{6}\right)^j - \frac{19}{34020} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{1}{2} h y_n^{(1)} - \frac{253}{10752} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{165}{1792}\left(\frac{1}{6}\right)^j - \frac{267}{17920}\left(\frac{1}{3}\right)^j + \frac{5}{128}\left(\frac{1}{2}\right)^j - \frac{363}{17920}\left(\frac{2}{3}\right)^j + \frac{57}{8960}\left(\frac{5}{6}\right)^j - \frac{47}{53760} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{2}{3} h y_n^{(1)} - \frac{272}{8505} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{376}{2835}\left(\frac{1}{6}\right)^j - \frac{2}{945}\left(\frac{1}{3}\right)^j + \frac{656}{8505}\left(\frac{1}{2}\right)^j - \frac{2}{81}\left(\frac{2}{3}\right)^j + \frac{8}{945}\left(\frac{5}{6}\right)^j - \frac{2}{1701} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{5}{6}\right)^j h^j}{j!} y_n^{(j)} - y_n - \frac{5}{6} h y_n^{(1)} - \frac{35225}{870912} h^2 y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{8375}{48384}\left(\frac{1}{6}\right)^j + \frac{3125}{290304}\left(\frac{1}{3}\right)^j + \frac{25625}{217728}\left(\frac{1}{2}\right)^j - \frac{625}{96768}\left(\frac{2}{3}\right)^j + \frac{275}{20736}\left(\frac{5}{6}\right) - \frac{1375}{870912} \right] \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j)} - y_n - h y_n^{(1)} - \frac{41}{840} h^2 y_n^2 - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{(j+2)} \left[\frac{3}{14}\left(\frac{1}{6}\right)^j + \frac{3}{140}\left(\frac{1}{3}\right)^j + \frac{17}{105}\left(\frac{1}{2}\right)^j + \frac{3}{280}\left(\frac{2}{3}\right) + \frac{3}{70}\left(\frac{5}{6}\right)^j \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{6}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{19087}{362880} h y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{2713}{15120}\left(\frac{1}{6}\right)^j - \frac{15487}{120960}\left(\frac{1}{3}\right)^j + \frac{293}{2835}\left(\frac{1}{2}\right)^j - \frac{6737}{120960}\left(\frac{2}{3}\right) + \frac{263}{15120}\left(\frac{5}{6}\right)^j - \frac{863}{362880} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{1139}{22680} h y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{47}{189}\left(\frac{1}{6}\right)^j + \frac{11}{7560}\left(\frac{1}{3}\right)^j + \frac{166}{2835}\left(\frac{1}{2}\right)^j - \frac{269}{7560}\left(\frac{2}{3}\right)^j + \frac{11}{945}\left(\frac{5}{6}\right)^j - \frac{37}{22680} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{137}{2688} h y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{27}{112}\left(\frac{1}{6}\right)^j - \frac{387}{4480}\left(\frac{1}{3}\right)^j + \frac{17}{105}\left(\frac{1}{2}\right)^j - \frac{243}{4480}\left(\frac{2}{3}\right)^j + \frac{9}{560}\left(\frac{5}{6}\right)^j - \frac{29}{13440} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{143}{2835} h y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{232}{945}\left(\frac{1}{6}\right)^j + \frac{64}{945}\left(\frac{1}{3}\right)^j + \frac{752}{2835}\left(\frac{1}{2}\right)^j + \frac{29}{945}\left(\frac{2}{3}\right) + \frac{8}{945}\left(\frac{5}{6}\right)^j - \frac{4}{2835} \right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{5}{6}\right)^j h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{3715}{72576} h y_n^2 - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{725}{3024}\left(\frac{1}{6}\right)^j + \frac{2125}{24192}\left(\frac{1}{3}\right)^j + \frac{125}{567}\left(\frac{1}{2}\right)^j + \frac{3875}{24192}\left(\frac{2}{3}\right)^j + \frac{235}{3024}\left(\frac{5}{6}\right)^j - \frac{275}{72576} \right] \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j+1)} - y_n^{(1)} - \frac{41}{840} h y_n^{(2)} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{(j+2)} \left[\frac{9}{35}\left(\frac{1}{6}\right)^j + \frac{9}{280}\left(\frac{1}{3}\right)^j + \frac{34}{105}\left(\frac{1}{2}\right)^j + \frac{9}{280}\left(\frac{2}{3}\right)^j + \frac{9}{35}\left(\frac{5}{6}\right)^j + \frac{41}{840} \right]
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.5.14)$$

and collecting terms in powers of h and y leads to

$$c_0 = \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{c}_1 = \begin{bmatrix} \frac{1}{6}-\frac{1}{6} \\ \frac{1}{3}-\frac{1}{3} \\ \frac{1}{2}-\frac{1}{2} \\ \frac{2}{3}-\frac{2}{3} \\ \frac{5}{6}-\frac{5}{6} \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\bar{c}_2 = \begin{bmatrix}
\left(\frac{1}{6}\right)^2 \frac{1}{2!} - \frac{28549}{4354560} - \left[\frac{275}{20736} - \frac{5717}{483840} + \frac{10621}{1088640} - \frac{7703}{1451520} + \frac{403}{241920} - \frac{199}{870912}\right] \\
\left(\frac{1}{3}\right)^2 \frac{1}{2!} - \frac{1027}{6804} - \left[\frac{97}{1890} - \frac{2}{81} + \frac{197}{8505} - \frac{97}{7560} + \frac{23}{5670} - \frac{19}{34020}\right] \\
\left(\frac{1}{2}\right)^2 \frac{1}{2!} - \frac{253}{10752} - \left[\frac{165}{1792} - \frac{267}{17920} + \frac{5}{128} - \frac{363}{17920} + \frac{57}{8960} - \frac{47}{53760}\right] \\
\left(\frac{2}{3}\right)^2 \frac{1}{2!} - \frac{272}{8505} - \left[\frac{376}{2835} - \frac{2}{945} + \frac{656}{8505} - \frac{2}{81} + \frac{8}{945} - \frac{2}{1701}\right] \\
\left(\frac{5}{6}\right)^2 \frac{1}{2!} - \frac{35225}{870912} - \left[\frac{8375}{48384} + \frac{3125}{290304} + \frac{25625}{217728} - \frac{625}{96768} + \frac{275}{20736} - \frac{1375}{870912}\right] \\
\frac{1}{2!} - \frac{41}{840} - \left[\frac{3}{14} + \frac{3}{140} + \frac{17}{105} + \frac{3}{280} + \frac{3}{70}\right] \\
\frac{1}{6} - \frac{19087}{362880} - \left[\frac{2713}{15120} - \frac{15487}{120960} + \frac{293}{2835} - \frac{6737}{120960} + \frac{263}{15120} - \frac{863}{362880}\right] \\
\frac{1}{3} - \frac{1139}{22680} - \left[\frac{47}{189} + \frac{11}{7560} + \frac{166}{2835} - \frac{269}{7560} + \frac{11}{945} - \frac{37}{22680}\right] \\
\frac{1}{2} - \frac{137}{2688} - \left[\frac{27}{112} + \frac{387}{4480} + \frac{17}{105} - \frac{243}{4480} + \frac{9}{560} - \frac{29}{13440}\right] \\
\frac{2}{3} - \frac{143}{2835} - \left[\frac{232}{945} + \frac{64}{945} + \frac{752}{2835} + \frac{29}{945} + \frac{8}{945} - \frac{4}{2835}\right] \\
\frac{5}{6} - \frac{3715}{72576} - \left[\frac{725}{3024} + \frac{2125}{24192} + \frac{125}{567} + \frac{3875}{24192} + \frac{235}{3024} - \frac{275}{72576}\right] \\
1 - \frac{41}{840} - \left[\frac{9}{35} + \frac{9}{280} + \frac{34}{105} + \frac{9}{280} + \frac{9}{35} + \frac{41}{840}\right]
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_3 = \begin{bmatrix}
\left(\frac{1}{6}\right)^3 \frac{1}{3!} - \left[\frac{275}{20796}\left(\frac{1}{6}\right) - \frac{5717}{483840}\left(\frac{1}{3}\right) + \frac{10621}{1088640}\left(\frac{1}{2}\right) - \frac{7703}{1451520}\left(\frac{2}{3}\right) + \frac{403}{241920}\left(\frac{5}{6}\right) - \frac{199}{870912}\right] \\
\left(\frac{1}{3}\right)^3 \frac{1}{3!} - \left[\frac{97}{1890}\left(\frac{1}{6}\right) - \frac{2}{81}\left(\frac{1}{3}\right) + \frac{197}{8505}\left(\frac{1}{2}\right) - \frac{97}{7560}\left(\frac{2}{3}\right) + \frac{26}{5670}\left(\frac{5}{6}\right) - \frac{19}{34020}\right] \\
\left(\frac{1}{2}\right)^3 \frac{1}{3!} - \left[\frac{165}{1792}\left(\frac{1}{6}\right) - \frac{267}{17920}\left(\frac{1}{3}\right) + \frac{5}{128}\left(\frac{1}{2}\right) - \frac{363}{17920}\left(\frac{2}{3}\right) + \frac{57}{8960}\left(\frac{5}{6}\right) - \frac{47}{53760}\right] \\
\left(\frac{2}{3}\right)^3 \frac{1}{3!} - \left[\frac{376}{2835}\left(\frac{1}{6}\right) - \frac{2}{945}\left(\frac{1}{3}\right) + \frac{656}{8505}\left(\frac{1}{2}\right) - \frac{2}{81}\left(\frac{2}{3}\right) + \frac{8}{945}\left(\frac{5}{6}\right) - \frac{2}{1701}\right] \\
\left(\frac{5}{6}\right)^3 \frac{1}{3!} - \left[\frac{8375}{48384}\left(\frac{1}{6}\right) + \frac{3125}{290304}\left(\frac{1}{3}\right) + \frac{25625}{217728}\left(\frac{1}{2}\right) - \frac{625}{96768}\left(\frac{2}{3}\right) + \frac{275}{20736}\left(\frac{5}{6}\right) - \frac{1375}{870912}\right] \\
\frac{1}{3!} - \left[\frac{3}{14}\left(\frac{1}{6}\right) + \frac{3}{140}\left(\frac{1}{3}\right) + \frac{17}{105}\left(\frac{1}{2}\right) + \frac{3}{280}\left(\frac{2}{3}\right) + \frac{3}{70}\left(\frac{5}{6}\right)\right] \\
\left(\frac{1}{6}\right)^2 \frac{1}{2!} - \left[\frac{2713}{15120}\left(\frac{1}{6}\right) - \frac{15487}{120960}\left(\frac{1}{3}\right) + \frac{293}{2835}\left(\frac{1}{2}\right) - \frac{6737}{120960}\left(\frac{2}{3}\right) + \frac{263}{15120}\left(\frac{5}{6}\right) - \frac{863}{362880}\right] \\
\left(\frac{1}{3}\right)^2 \frac{1}{2!} - \left[\frac{47}{189}\left(\frac{1}{6}\right) + \frac{11}{7560}\left(\frac{1}{3}\right) + \frac{166}{2835}\left(\frac{1}{2}\right) - \frac{269}{7560}\left(\frac{2}{3}\right) + \frac{11}{945}\left(\frac{5}{6}\right) - \frac{37}{22680}\right] \\
\left(\frac{1}{2}\right)^2 \frac{1}{2!} - \left[\frac{27}{112}\left(\frac{1}{6}\right) - \frac{387}{4480}\left(\frac{1}{3}\right) + \frac{17}{105}\left(\frac{1}{2}\right) - \frac{243}{4480}\left(\frac{2}{3}\right) + \frac{9}{560}\left(\frac{5}{6}\right) - \frac{29}{13440}\right] \\
\left(\frac{2}{3}\right)^2 \frac{1}{2!} - \left[\frac{232}{945}\left(\frac{1}{6}\right) + \frac{64}{945}\left(\frac{1}{3}\right) + \frac{752}{2835}\left(\frac{1}{2}\right) - + \frac{29}{945}\left(\frac{2}{3}\right) + \frac{98}{945}\left(\frac{5}{6}\right) - \frac{4}{2835}\right] \\
\left(\frac{5}{6}\right)^2 \frac{1}{3!} - \left[\frac{725}{3024}\left(\frac{1}{6}\right) + \frac{2125}{24192}\left(\frac{1}{3}\right) + \frac{125}{567}\left(\frac{1}{2}\right) + \frac{3875}{24192}\left(\frac{2}{3}\right) + \frac{235}{3024}\left(\frac{5}{6}\right) - \frac{275}{72576}\right] \\
\frac{1}{2!} - \left[\frac{9}{35}\left(\frac{1}{6}\right) + \frac{9}{280}\left(\frac{1}{3}\right) + \frac{34}{105}\left(\frac{1}{2}\right) + \frac{9}{280}\left(\frac{2}{3}\right) + \frac{9}{35}\left(\frac{5}{6}\right) + \frac{41}{840}\right]
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_4 \;=\; \begin{bmatrix}
\left(\frac{1}{6}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{275}{20736}\left(\frac{1}{6}\right)^2 - \frac{5717}{483840}\left(\frac{1}{3}\right)^2 + \frac{10621}{1088640}\left(\frac{1}{2}\right)^2 - \frac{7703}{1451520}\left(\frac{2}{3}\right)^2 + \frac{403}{241920}\left(\frac{5}{6}\right)^2 - \frac{199}{870912}\right] \\
\left(\frac{1}{3}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{97}{1890}\left(\frac{1}{6}\right)^2 - \frac{2}{81}\left(\frac{1}{3}\right)^2 + \frac{197}{8505}\left(\frac{1}{2}\right)^2 - \frac{97}{7560}\left(\frac{2}{3}\right)^2 + \frac{26}{5670}\left(\frac{5}{6}\right)^2 - \frac{19}{34020}\right] \\
\left(\frac{1}{2}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{165}{1792}\left(\frac{1}{6}\right)^2 - \frac{267}{17920}\left(\frac{1}{3}\right)^2 + \frac{5}{128}\left(\frac{1}{2}\right)^2 - \frac{363}{17920}\left(\frac{2}{3}\right)^2 + \frac{57}{8960}\left(\frac{5}{6}\right)^2 - \frac{47}{53760}\right] \\
\left(\frac{2}{3}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{376}{2835}\left(\frac{1}{6}\right)^2 - \frac{2}{945}\left(\frac{1}{3}\right)^2 + \frac{656}{8505}\left(\frac{1}{2}\right)^2 - \frac{2}{81}\left(\frac{2}{3}\right)^2 + \frac{8}{945}\left(\frac{5}{6}\right)^2 - \frac{2}{1701}\right] \\
\left(\frac{5}{6}\right)^4 \frac{1}{4!} - \frac{1}{2!}\left[\frac{8375}{48384}\left(\frac{1}{6}\right)^2 + \frac{3125}{290304}\left(\frac{1}{3}\right)^2 + \frac{25625}{217728}\left(\frac{1}{2}\right)^2 - \frac{625}{96768}\left(\frac{2}{3}\right)^2 + \frac{275}{20736}\left(\frac{5}{6}\right)^2 - \frac{1375}{870912}\right] \\
\frac{1}{4!} - \frac{1}{2!}\left[\frac{3}{14}\left(\frac{1}{6}\right)^2 + \frac{3}{140}\left(\frac{1}{3}\right)^2 + \frac{17}{105}\left(\frac{1}{2}\right)^2 + \frac{3}{280}\left(\frac{2}{3}\right)^2 + \frac{3}{70}\left(\frac{5}{6}\right)^2\right] \\
\left(\frac{1}{6}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{2713}{15120}\left(\frac{1}{6}\right)^2 - \frac{15487}{120960}\left(\frac{1}{3}\right)^2 + \frac{293}{2835}\left(\frac{1}{2}\right)^2 - \frac{6737}{120960}\left(\frac{2}{3}\right)^2 + \frac{263}{150}\left(\frac{5}{6}\right)^2 - \frac{863}{362880}\right] \\
\left(\frac{1}{3}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{47}{189}\left(\frac{1}{6}\right)^2 + \frac{11}{7560}\left(\frac{1}{3}\right)^2 + \frac{166}{2835}\left(\frac{1}{2}\right)^2 - \frac{269}{7560}\left(\frac{2}{3}\right)^2 + \frac{11}{945}\left(\frac{5}{6}\right)^2 - \frac{37}{22680}\right] \\
\left(\frac{1}{2}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{27}{112}\left(\frac{1}{6}\right)^2 + \frac{387}{4480}\left(\frac{1}{3}\right)^2 + \frac{17}{105}\left(\frac{1}{2}\right)^2 - \frac{243}{4480}\left(\frac{2}{3}\right)^2 + \frac{9}{560}\left(\frac{5}{6}\right)^2 - \frac{29}{13440}\right] \\
\left(\frac{2}{3}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{232}{945}\left(\frac{1}{6}\right)^2 + \frac{64}{945}\left(\frac{1}{3}\right)^2 + \frac{752}{2835}\left(\frac{1}{2}\right)^2 + \frac{29}{945}\left(\frac{2}{3}\right)^2 + \frac{8}{945}\left(\frac{5}{6}\right)^2 - \frac{4}{2835}\right] \\
\left(\frac{5}{6}\right)^3 \frac{1}{3!} - \frac{1}{2!}\left[\frac{725}{3024}\left(\frac{1}{6}\right)^2 + \frac{2125}{24192}\left(\frac{1}{3}\right)^2 + \frac{125}{567}\left(\frac{1}{2}\right)^2 + \frac{3875}{24192}\left(\frac{2}{3}\right)^2 + \frac{235}{3024}\left(\frac{5}{6}\right)^2 - \frac{275}{72576}\right] \\
\frac{1}{3!} - \frac{1}{2!}\left[\frac{9}{35}\left(\frac{1}{6}\right)^2 + \frac{9}{280}\left(\frac{1}{3}\right)^2 + \frac{34}{105}\left(\frac{1}{2}\right)^2 + \frac{9}{280}\left(\frac{2}{3}\right)^2 + \frac{9}{35}\left(\frac{5}{6}\right)^2 - \frac{41}{840}\right]
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_5 = \begin{bmatrix}
\left(\frac{1}{6}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{275}{20736}\left(\frac{1}{6}\right)^3 - \frac{5717}{483840}\left(\frac{1}{3}\right)^3 + \frac{10621}{1088640}\left(\frac{1}{2}\right)^3 - \frac{7703}{1451520}\left(\frac{2}{3}\right)^3 + \frac{403}{241920}\left(\frac{5}{6}\right)^3 - \frac{199}{870912}\right] \\
\left(\frac{1}{3}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{97}{1890}\left(\frac{1}{6}\right)^3 - \frac{2}{81}\left(\frac{1}{3}\right)^3 + \frac{197}{8505}\left(\frac{1}{2}\right)^3 - \frac{97}{7560}\left(\frac{2}{3}\right)^3 + \frac{23}{5670}\left(\frac{5}{6}\right)^3 - \frac{19}{34020}\right] \\
\left(\frac{1}{2}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{165}{1792}\left(\frac{1}{6}\right)^3 - \frac{267}{17920}\left(\frac{1}{3}\right)^3 + \frac{5}{128}\left(\frac{1}{2}\right)^3 - \frac{363}{17920}\left(\frac{2}{3}\right)^3 + \frac{57}{8960}\left(\frac{5}{6}\right)^3 - \frac{47}{53760}\right] \\
\left(\frac{2}{3}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{376}{2835}\left(\frac{1}{6}\right)^3 - \frac{2}{945}\left(\frac{1}{3}\right)^3 + \frac{656}{8505}\left(\frac{1}{2}\right)^2 - \frac{2}{81}\left(\frac{2}{3}\right)^3 + \frac{8}{945}\left(\frac{5}{6}\right)^3 - \frac{2}{1701}\right] \\
\left(\frac{5}{6}\right)^5 \frac{1}{5!} - \frac{1}{3!}\left[\frac{8375}{48384}\left(\frac{1}{6}\right)^3 + \frac{3125}{290304}\left(\frac{1}{3}\right)^3 + \frac{25625}{217728}\left(\frac{1}{2}\right)^3 - \frac{625}{96768}\left(\frac{2}{3}\right)^3 + \frac{275}{20736}\left(\frac{5}{6}\right)^2 - \frac{1375}{870912}\right] \\
\frac{1}{5!} - \frac{1}{3!}\left[\frac{3}{14}\left(\frac{1}{6}\right)^3 + \frac{3}{140}\left(\frac{1}{3}\right)^3 + \frac{17}{105}\left(\frac{1}{2}\right)^3 + \frac{3}{280}\left(\frac{2}{3}\right)^3 + \frac{3}{70}\left(\frac{5}{6}\right)^3\right] \\
\left(\frac{1}{6}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{2713}{15120}\left(\frac{1}{6}\right)^3 - \frac{15487}{120960}\left(\frac{1}{3}\right)^3 + \frac{293}{2835}\left(\frac{1}{2}\right)^3 - \frac{6737}{120960}\left(\frac{2}{3}\right)^3 + \frac{265}{15120}\left(\frac{5}{6}\right)^3 - \frac{863}{362880}\right] \\
\left(\frac{1}{3}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{47}{189}\left(\frac{1}{6}\right)^3 + \frac{11}{7560}\left(\frac{1}{3}\right)^3 + \frac{166}{2835}\left(\frac{1}{2}\right)^3 - \frac{269}{7560}\left(\frac{2}{3}\right)^3 + \frac{11}{945}\left(\frac{5}{6}\right)^3 - \frac{37}{22680}\right] \\
\left(\frac{1}{2}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{27}{112}\left(\frac{1}{6}\right)^3 + \frac{387}{4480}\left(\frac{1}{3}\right)^3 + \frac{17}{105}\left(\frac{1}{2}\right)^3 - \frac{243}{4480}\left(\frac{2}{3}\right)^3 + \frac{9}{560}\left(\frac{5}{6}\right)^3 - \frac{29}{13440}\right] \\
\left(\frac{2}{3}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{232}{945}\left(\frac{1}{6}\right)^3 + \frac{64}{945}\left(\frac{1}{3}\right)^3 + \frac{752}{2835}\left(\frac{1}{2}\right)^3 - \frac{29}{945}\left(\frac{2}{3}\right)^3 + \frac{8}{945}\left(\frac{5}{6}\right)^3 - \frac{4}{2835}\right] \\
\left(\frac{5}{6}\right)^4 \frac{1}{4!} - \frac{1}{3!}\left[\frac{725}{3024}\left(\frac{1}{6}\right)^3 + \frac{2125}{24192}\left(\frac{1}{3}\right)^3 + \frac{125}{567}\left(\frac{1}{2}\right)^3 + \frac{3875}{24192}\left(\frac{2}{3}\right)^3 + \frac{235}{3024}\left(\frac{5}{6}\right)^3 - \frac{275}{72576}\right] \\
\frac{1}{4!} - \frac{1}{3!}\left[\frac{9}{35}\left(\frac{1}{6}\right)^3 + \frac{9}{280}\left(\frac{1}{3}\right)^3 + \frac{34}{105}\left(\frac{1}{2}\right)^3 + \frac{9}{280}\left(\frac{2}{3}\right)^3 + \frac{9}{354}\left(\frac{5}{6}\right)^3 - \frac{41}{840}\right]
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_6 = \begin{bmatrix}
\left(\tfrac{1}{6}\right)^6 \tfrac{1}{6!} - \tfrac{1}{4!}\left[\tfrac{275}{20736}\left(\tfrac{1}{6}\right)^4 - \tfrac{5717}{483840}\left(\tfrac{1}{3}\right)^3 + \tfrac{10621}{1088640}\left(\tfrac{1}{2}\right)^3 - \tfrac{7703}{1451520}\left(\tfrac{2}{3}\right)^3 + \tfrac{403}{241920}\left(\tfrac{5}{6}\right)^3 - \tfrac{199}{870912}\right] \\
\left(\tfrac{1}{3}\right)^6 \tfrac{1}{6!} - \tfrac{1}{4!}\left[\tfrac{97}{1890}\left(\tfrac{1}{6}\right)^4 - \tfrac{2}{81}\left(\tfrac{1}{3}\right)^4 + \tfrac{197}{8505}\left(\tfrac{1}{2}\right)^4 - \tfrac{97}{7560}\left(\tfrac{2}{3}\right)^4 + \tfrac{23}{5670}\left(\tfrac{5}{6}\right)^4 - \tfrac{19}{34020}\right] \\
\left(\tfrac{1}{2}\right)^6 \tfrac{1}{6!} - \tfrac{1}{4!}\left[\tfrac{165}{1792}\left(\tfrac{1}{6}\right)^4 - \tfrac{267}{17920}\left(\tfrac{1}{3}\right)^4 + \tfrac{5}{128}\left(\tfrac{1}{2}\right)^4 - \tfrac{363}{17920}\left(\tfrac{2}{3}\right)^4 + \tfrac{57}{8960}\left(\tfrac{5}{6}\right)^4 - \tfrac{47}{53760}\right] \\
\left(\tfrac{2}{3}\right)^6 \tfrac{1}{6!} - \tfrac{1}{4!}\left[\tfrac{376}{2835}\left(\tfrac{1}{6}\right)^4 - \tfrac{2}{945}\left(\tfrac{1}{3}\right)^4 + \tfrac{656}{8505}\left(\tfrac{1}{2}\right)^4 - \tfrac{2}{81}\left(\tfrac{2}{3}\right)^4 + \tfrac{8}{945}\left(\tfrac{5}{6}\right)^4 - \tfrac{2}{1701}\right] \\
\left(\tfrac{5}{6}\right)^6 \tfrac{1}{6!} - \tfrac{1}{4!}\left[\tfrac{8375}{48384}\left(\tfrac{1}{6}\right)^4 + \tfrac{3125}{290304}\left(\tfrac{1}{3}\right)^4 + \tfrac{25625}{217728}\left(\tfrac{1}{2}\right)^4 - \tfrac{625}{96768}\left(\tfrac{2}{3}\right)^4 + \tfrac{275}{20736}\left(\tfrac{5}{6}\right)^4 - \tfrac{1375}{870912}\right] \\
\tfrac{1}{6!} - \tfrac{1}{4!}\left[\tfrac{3}{14}\left(\tfrac{1}{6}\right)^4 + \tfrac{3}{140}\left(\tfrac{1}{3}\right)^4 + \tfrac{17}{105}\left(\tfrac{1}{2}\right)^4 + \tfrac{3}{280}\left(\tfrac{2}{3}\right)^4 + \tfrac{3}{70}\left(\tfrac{5}{6}\right)^4\right] \\
\left(\tfrac{1}{6}\right)^5 \tfrac{1}{5!} - \tfrac{1}{4!}\left[\tfrac{2713}{15120}\left(\tfrac{1}{6}\right)^4 - \tfrac{15487}{120960}\left(\tfrac{1}{3}\right)^4 + \tfrac{293}{2835}\left(\tfrac{1}{2}\right)^4 - \tfrac{6737}{120960}\left(\tfrac{2}{3}\right)^4 + \tfrac{265}{15120}\left(\tfrac{5}{6}\right)^4 - \tfrac{863}{362880}\right] \\
\left(\tfrac{1}{3}\right)^5 \tfrac{1}{5!} - \tfrac{1}{4!}\left[\tfrac{47}{189}\left(\tfrac{1}{6}\right)^4 + \tfrac{11}{7560}\left(\tfrac{1}{3}\right)^4 + \tfrac{166}{2835}\left(\tfrac{1}{2}\right)^4 - \tfrac{269}{7560}\left(\tfrac{2}{3}\right)^4 + \tfrac{11}{945}\left(\tfrac{5}{6}\right)^4 - \tfrac{37}{22680}\right] \\
\left(\tfrac{1}{2}\right)^5 \tfrac{1}{5!} - \tfrac{1}{4!}\left[\tfrac{27}{112}\left(\tfrac{1}{6}\right)^4 + \tfrac{387}{4480}\left(\tfrac{1}{3}\right)^4 + \tfrac{17}{105}\left(\tfrac{1}{2}\right)^4 - \tfrac{243}{4480}\left(\tfrac{2}{3}\right)^4 + \tfrac{9}{560}\left(\tfrac{5}{6}\right)^4 - \tfrac{29}{13440}\right] \\
\left(\tfrac{2}{3}\right)^5 \tfrac{1}{5!} - \tfrac{1}{4!}\left[\tfrac{232}{945}\left(\tfrac{1}{6}\right)^4 + \tfrac{64}{945}\left(\tfrac{1}{3}\right)^4 + \tfrac{752}{2835}\left(\tfrac{1}{2}\right)^4 + \tfrac{29}{945}\left(\tfrac{2}{3}\right)^4 + \tfrac{8}{945}\left(\tfrac{5}{6}\right)^4 - \tfrac{4}{2835}\right] \\
\left(\tfrac{5}{6}\right)^5 \tfrac{1}{5!} - \tfrac{1}{4!}\left[\tfrac{725}{3024}\left(\tfrac{1}{6}\right)^4 + \tfrac{2125}{24192}\left(\tfrac{1}{3}\right)^4 + \tfrac{125}{567}\left(\tfrac{1}{2}\right)^4 + \tfrac{3875}{24192}\left(\tfrac{2}{3}\right)^4 + \tfrac{235}{3024}\left(\tfrac{5}{6}\right)^4 - \tfrac{275}{72576}\right] \\
\tfrac{1}{5!} - \tfrac{1}{4!}\left[\tfrac{9}{35}\left(\tfrac{1}{6}\right)^4 + \tfrac{9}{280}\left(\tfrac{1}{3}\right)^4 + \tfrac{34}{105}\left(\tfrac{1}{2}\right)^4 + \tfrac{9}{280}\left(\tfrac{2}{3}\right)^4 + \tfrac{9}{35}\left(\tfrac{5}{6}\right)^4 - \tfrac{41}{840}\right]
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_7 = \begin{bmatrix}
\left(\frac{1}{6}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{275}{20736}\left(\frac{1}{6}\right)^5 - \frac{5717}{483840}\left(\frac{1}{3}\right)^5 + \frac{10621}{1088640}\left(\frac{1}{2}\right)^5 - \frac{7703}{1451520}\left(\frac{2}{3}\right)^5 + \frac{403}{241920}\left(\frac{5}{6}\right)^5 - \frac{199}{870912}\right] \\
\left(\frac{1}{3}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{97}{1890}\left(\frac{1}{6}\right)^5 - \frac{2}{81}\left(\frac{1}{3}\right)^5 + \frac{197}{8505}\left(\frac{1}{2}\right)^5 - \frac{97}{7560}\left(\frac{2}{3}\right)^5 + \frac{23}{5670}\left(\frac{5}{6}\right)^5 - \frac{19}{34020}\right] \\
\left(\frac{1}{2}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{165}{1792}\left(\frac{1}{6}\right)^5 - \frac{267}{17920}\left(\frac{1}{3}\right)^5 + \frac{5}{128}\left(\frac{1}{2}\right)^5 - \frac{363}{17920}\left(\frac{2}{3}\right)^5 + \frac{57}{8960}\left(\frac{5}{6}\right)^5 - \frac{47}{53760}\right] \\
\left(\frac{2}{3}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{376}{48384}\left(\frac{1}{6}\right)^5 - \frac{2}{945}\left(\frac{1}{3}\right)^5 + \frac{656}{8505}\left(\frac{1}{2}\right)^5 - \frac{2}{81}\left(\frac{2}{3}\right)^5 + \frac{8}{945}\left(\frac{5}{6}\right)^5 - \frac{2}{1701}\right] \\
\left(\frac{5}{6}\right)^7 \frac{1}{7!} - \frac{1}{5!}\left[\frac{8375}{48384}\left(\frac{1}{6}\right)^5 + \frac{3125}{290304}\left(\frac{1}{3}\right)^5 + \frac{25625}{217728}\left(\frac{1}{2}\right)^5 - \frac{625}{96768}\left(\frac{2}{3}\right)^5 + \frac{275}{20736}\left(\frac{5}{6}\right)^5 - \frac{1375}{870912}\right] \\
\frac{1}{7!} - \frac{1}{5!}\left[\frac{3}{14}\left(\frac{1}{6}\right)^5 + \frac{3}{140}\left(\frac{1}{3}\right)^5 + \frac{17}{105}\left(\frac{1}{2}\right)^5 + \frac{3}{280}\left(\frac{2}{3}\right)^5 + \frac{3}{70}\left(\frac{5}{6}\right)^5\right] \\
\left(\frac{1}{6}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{2713}{15120}\left(\frac{1}{6}\right)^5 - \frac{15487}{120960}\left(\frac{1}{3}\right)^5 + \frac{293}{2835}\left(\frac{1}{2}\right)^5 - \frac{6737}{120960}\left(\frac{2}{3}\right)^5 + \frac{265}{15120}\left(\frac{5}{6}\right)^5 - \frac{863}{362880}\right] \\
\left(\frac{1}{3}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{47}{189}\left(\frac{1}{6}\right)^5 + \frac{11}{7560}\left(\frac{1}{3}\right)^5 + \frac{166}{2835}\left(\frac{1}{2}\right)^5 - \frac{269}{7560}\left(\frac{2}{3}\right)^5 + \frac{11}{945}\left(\frac{5}{6}\right)^5 - \frac{37}{22680}\right] \\
\left(\frac{1}{2}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{27}{112}\left(\frac{1}{6}\right)^5 + \frac{387}{4480}\left(\frac{1}{3}\right)^5 + \frac{17}{105}\left(\frac{1}{2}\right)^5 - \frac{243}{4480}\left(\frac{2}{3}\right)^5 + \frac{9}{560}\left(\frac{5}{6}\right)^5 - \frac{29}{13440}\right] \\
\left(\frac{2}{3}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{232}{945}\left(\frac{1}{6}\right)^5 + \frac{64}{945}\left(\frac{1}{3}\right)^5 + \frac{752}{2835}\left(\frac{1}{2}\right)^5 + \frac{29}{945}\left(\frac{2}{3}\right)^5 + \frac{8}{945}\left(\frac{5}{6}\right)^5 - \frac{4}{2835}\right] \\
\left(\frac{5}{6}\right)^6 \frac{1}{6!} - \frac{1}{5!}\left[\frac{725}{3024}\left(\frac{1}{6}\right)^5 + \frac{2125}{24192}\left(\frac{1}{3}\right)^5 + \frac{125}{567}\left(\frac{1}{2}\right)^5 + \frac{3875}{24192}\left(\frac{2}{3}\right)^5 + \frac{235}{3024}\left(\frac{5}{6}\right)^5 - \frac{275}{72576}\right] \\
\frac{1}{6!} - \frac{1}{5!}\left[\frac{9}{35}\left(\frac{1}{6}\right)^5 + \frac{9}{280}\left(\frac{1}{3}\right)^5 + \frac{34}{105}\left(\frac{1}{2}\right)^5 + \frac{9}{280}\left(\frac{2}{3}\right)^5 + \frac{9}{35}\left(\frac{5}{6}\right)^5 - \frac{41}{840}\right]
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_8 = \begin{bmatrix}
\left(\frac{1}{6}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{275}{20736}\left(\frac{1}{6}\right)^6 - \frac{5717}{483840}\left(\frac{1}{3}\right)^6 + \frac{10621}{1088640}\left(\frac{1}{2}\right)^6 - \frac{7703}{1451520}\left(\frac{2}{3}\right)^6 + \frac{403}{241920}\left(\frac{5}{6}\right)^6 - \frac{199}{870912}\right] \\
\left(\frac{1}{3}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{97}{1890}\left(\frac{1}{6}\right)^6 - \frac{2}{81}\left(\frac{1}{3}\right)^6 + \frac{197}{8505}\left(\frac{1}{2}\right)^6 - \frac{97}{7560}\left(\frac{2}{3}\right)^6 + \frac{23}{5670}\left(\frac{5}{6}\right)^6 - \frac{19}{34020}\right] \\
\left(\frac{1}{2}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{165}{1792}\left(\frac{1}{6}\right)^6 - \frac{267}{17920}\left(\frac{1}{3}\right)^6 + \frac{5}{128}\left(\frac{1}{2}\right)^6 - \frac{363}{17920}\left(\frac{2}{3}\right)^6 + \frac{57}{8960}\left(\frac{5}{6}\right)^6 - \frac{47}{53760}\right] \\
\left(\frac{2}{3}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{376}{48384}\left(\frac{1}{6}\right)^6 - \frac{2}{945}\left(\frac{1}{3}\right)^6 + \frac{656}{8505}\left(\frac{1}{2}\right)^6 - \frac{2}{81}\left(\frac{2}{3}\right)^6 + \frac{8}{945}\left(\frac{5}{6}\right)^6 - \frac{2}{1701}\right] \\
\left(\frac{5}{6}\right)^8 \frac{1}{8!} - \frac{1}{6!}\left[\frac{8375}{48384}\left(\frac{1}{6}\right)^6 + \frac{3125}{290304}\left(\frac{1}{3}\right)^6 + \frac{25625}{217728}\left(\frac{1}{2}\right)^6 - \frac{625}{96768}\left(\frac{2}{3}\right)^6 + \frac{275}{20736}\left(\frac{5}{6}\right)^6 - \frac{1375}{870912}\right] \\
\frac{1}{8!} - \frac{1}{6!}\left[\frac{3}{14}\left(\frac{1}{6}\right)^6 + \frac{3}{140}\left(\frac{1}{3}\right)^6 + \frac{17}{105}\left(\frac{1}{2}\right)^6 + \frac{3}{280}\left(\frac{2}{3}\right)^6 + \frac{3}{70}\left(\frac{5}{6}\right)^6\right] \\
\left(\frac{1}{6}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{2713}{15120}\left(\frac{1}{6}\right)^6 - \frac{15487}{120960}\left(\frac{1}{3}\right)^6 + \frac{293}{2835}\left(\frac{1}{2}\right)^6 - \frac{6737}{120960}\left(\frac{2}{3}\right)^6 + \frac{265}{15120}\left(\frac{5}{6}\right)^6 - \frac{863}{362880}\right] \\
\left(\frac{1}{3}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{47}{189}\left(\frac{1}{6}\right)^6 + \frac{11}{7560}\left(\frac{1}{3}\right)^6 + \frac{166}{2835}\left(\frac{1}{2}\right)^6 - \frac{269}{7560}\left(\frac{2}{3}\right)^6 + \frac{11}{945}\left(\frac{5}{6}\right)^6 - \frac{37}{22680}\right] \\
\left(\frac{1}{2}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{27}{112}\left(\frac{1}{6}\right)^6 + \frac{387}{4480}\left(\frac{1}{3}\right)^6 + \frac{17}{105}\left(\frac{1}{2}\right)^6 - \frac{243}{4480}\left(\frac{2}{3}\right)^6 + \frac{9}{560}\left(\frac{5}{6}\right)^6 - \frac{29}{13440}\right] \\
\left(\frac{2}{3}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{232}{945}\left(\frac{1}{6}\right)^6 + \frac{64}{945}\left(\frac{1}{3}\right)^6 + \frac{752}{2835}\left(\frac{1}{2}\right)^6 + \frac{29}{945}\left(\frac{2}{3}\right)^6 + \frac{8}{945}\left(\frac{5}{6}\right)^6 - \frac{4}{2835}\right] \\
\left(\frac{5}{6}\right)^7 \frac{1}{7!} - \frac{1}{6!}\left[\frac{725}{3024}\left(\frac{1}{6}\right)^6 + \frac{2125}{24192}\left(\frac{1}{3}\right)^6 + \frac{125}{567}\left(\frac{1}{2}\right)^6 + \frac{3875}{24192}\left(\frac{2}{3}\right)^6 + \frac{235}{3024}\left(\frac{5}{6}\right)^6 - \frac{275}{72576}\right] \\
\frac{1}{7!} - \frac{1}{6!}\left[\frac{9}{35}\left(\frac{1}{6}\right)^6 + \frac{9}{280}\left(\frac{1}{3}\right)^6 + \frac{34}{105}\left(\frac{1}{2}\right)^6 + \frac{9}{280}\left(\frac{2}{3}\right)^6 + \frac{9}{35}\left(\frac{5}{6}\right)^6 - \frac{41}{840}\right]
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\bar{c}_9 = \begin{bmatrix}
\left(\frac{1}{6}\right)^9 \frac{1}{9!} - \frac{1}{7!}\Big[\frac{275}{20736}\left(\frac{1}{6}\right)^7 - \frac{5717}{483840}\left(\frac{1}{3}\right)^7 + \frac{10621}{1088640}\left(\frac{1}{2}\right)^7 - \frac{7703}{1451520}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{403}{241920}\left(\frac{5}{6}\right)^7 - \frac{199}{870912}\Big] \\
\left(\frac{1}{3}\right)^9 \frac{1}{9!} - \frac{1}{7!}\Big[\frac{97}{1890}\left(\frac{1}{6}\right)^7 - \frac{2}{81}\left(\frac{1}{3}\right)^7 + \frac{197}{8505}\left(\frac{1}{2}\right)^7 - \frac{97}{7560}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{23}{5670}\left(\frac{5}{6}\right)^7 - \frac{19}{34020}\Big] \\
\left(\frac{1}{2}\right)^9 \frac{1}{9!} - \frac{1}{7!}\Big[\frac{165}{1792}\left(\frac{1}{6}\right)^7 - \frac{267}{17920}\left(\frac{1}{3}\right)^7 + \frac{5}{128}\left(\frac{1}{2}\right)^7 - \frac{363}{17920}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{57}{8960}\left(\frac{5}{6}\right)^7 - \frac{47}{53760}\Big] \\
\left(\frac{2}{3}\right)^9 \frac{1}{9!} - \frac{1}{7!}\Big[\frac{376}{48384}\left(\frac{1}{6}\right)^7 - \frac{2}{945}\left(\frac{1}{3}\right)^7 + \frac{656}{8505}\left(\frac{1}{2}\right)^7 - \frac{2}{81}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{8}{945}\left(\frac{5}{6}\right)^7 - \frac{2}{1701}\Big] \\
\left(\frac{5}{6}\right)^9 \frac{1}{9!} - \frac{1}{7!}\Big[\frac{8375}{48384}\left(\frac{1}{6}\right)^7 + \frac{3125}{290304}\left(\frac{1}{3}\right)^7 + \frac{25625}{217728}\left(\frac{1}{2}\right)^7 - \frac{625}{96768}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{275}{20736}\left(\frac{5}{6}\right)^7 - \frac{1375}{870912}\Big] \\
\frac{1}{9!} - \frac{1}{7!}\Big[\frac{3}{14}\left(\frac{1}{6}\right)^7 + \frac{3}{140}\left(\frac{1}{3}\right)^7 + \frac{17}{105}\left(\frac{1}{2}\right)^7 + \frac{3}{280}\left(\frac{2}{3}\right)^7 + \frac{3}{70}\left(\frac{5}{6}\right)^7\Big] \\
\left(\frac{1}{6}\right)^8 \frac{1}{8!} - \frac{1}{7!}\Big[\frac{2713}{15120}\left(\frac{1}{6}\right)^7 - \frac{15487}{120960}\left(\frac{1}{3}\right)^7 + \frac{293}{2835}\left(\frac{1}{2}\right)^7 - \frac{6737}{120960}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{265}{15120}\left(\frac{5}{6}\right)^7 - \frac{863}{362880}\Big] \\
\left(\frac{1}{3}\right)^8 \frac{1}{8!} - \frac{1}{7!}\Big[\frac{47}{189}\left(\frac{1}{6}\right)^7 + \frac{11}{7560}\left(\frac{1}{3}\right)^7 + \frac{166}{2835}\left(\frac{1}{2}\right)^7 - \frac{269}{7560}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{11}{945}\left(\frac{5}{6}\right)^7 - \frac{37}{22680}\Big] \\
\left(\frac{1}{2}\right)^8 \frac{1}{8!} - \frac{1}{7!}\Big[\frac{27}{112}\left(\frac{1}{6}\right)^7 + \frac{387}{4480}\left(\frac{1}{3}\right)^7 + \frac{17}{105}\left(\frac{1}{2}\right)^7 - \frac{243}{4480}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{9}{560}\left(\frac{5}{6}\right)^7 - \frac{29}{13440}\Big] \\
\left(\frac{2}{3}\right)^8 \frac{1}{8!} - \frac{1}{7!}\Big[\frac{232}{945}\left(\frac{1}{6}\right)^7 + \frac{64}{945}\left(\frac{1}{3}\right)^7 + \frac{752}{2835}\left(\frac{1}{2}\right)^7 + \frac{29}{945}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{8}{945}\left(\frac{5}{6}\right)^7 - \frac{4}{2835}\Big] \\
\left(\frac{5}{6}\right)^8 \frac{1}{8!} - \frac{1}{7!}\Big[\frac{725}{3024}\left(\frac{1}{6}\right)^7 + \frac{2125}{24192}\left(\frac{1}{3}\right)^7 + \frac{125}{567}\left(\frac{1}{2}\right)^7 + \frac{3875}{24192}\left(\frac{2}{3}\right)^7 \\
\qquad + \frac{235}{3024}\left(\frac{5}{6}\right)^7 - \frac{275}{72576}\Big] \\
\frac{1}{8!} - \frac{1}{7!}\Big[\frac{9}{35}\left(\frac{1}{6}\right)^7 + \frac{9}{280}\left(\frac{1}{3}\right)^7 + \frac{34}{105}\left(\frac{1}{2}\right)^7 + \frac{9}{280}\left(\frac{2}{3}\right)^7 + \frac{9}{354}\left(\frac{5}{6}\right)^7 + \frac{41}{840}\Big]
\end{bmatrix}
$$

$$= \begin{bmatrix} \frac{6031}{9142485811200} \\ \frac{233}{142851340800} \\ \frac{1}{391910400} \\ \frac{455743}{127994801356800} \\ \frac{10055}{519792054272} \\ \frac{1}{195955200} \\ \frac{475}{853298675712} \\ \frac{1}{198404640} \\ \frac{1}{167215104} \\ \frac{1}{198404640} \\ \frac{275}{40633270272} \\ 0 \end{bmatrix}$$

Thus, the block method (3.3.64) has order $p = (7,7,7,7,7,7,7,7,7,7,7,7)^T$ and error constant $c_{p+2} = \big(\frac{6031}{914248581120}, \frac{233}{142851340800}, \frac{1}{391910400}, \frac{455743}{127994801356800}, \frac{10055}{519792054272}, \frac{1}{195955200}, \frac{475}{853298675712}, \frac{1}{198404640}, \frac{1}{167215104}, \frac{1}{198404640}, \frac{275}{40633270272}, 0\big)^T$

4.6.2 Consistency

The block method (3.3.64) has order $p = (7,7,7,7,7,7,7,7,7,7,7,7)^T > 1$, therefore, by condition (i) in definition (4.1.6) it is consistent. Following the consistency of the block method associated with the main method (3.3.60e), the consistency of this method is shown as follows by conditions (i) - (iv) in definition (4.1.6);

Conditon(i)

The main method (3.3.60e) has order $p = 7 > 1$, which it satisfies condition (i) in definition (4.1.6).

Condition (ii)

The first and second characteristic polynomials of method (3.3.60e) are given respectively by

$$\rho(z) = z - 2z^{5/6} + z^{2/3} \tag{4.5.15}$$

and

$$\sigma(z) = -\left(\frac{-4315z - 53994z^{5/6} + 2307z^{2/3} - 7948z^{1/2} + 4827z^{1/3} - 1578z^{1/6} + 221}{2177280}\right) \tag{4.5.16}$$

From (4.5.15), $\alpha_{2/3} = 1$, $\alpha_{5/6} = -2$ and $\alpha_1 = 1$ thus

$$\sum_j \alpha_j = 1 - 2 + 1 = 0, \quad j = \frac{2}{3},\ \frac{5}{6},\ 1$$

Condition (iii)

$$\begin{aligned} \rho'(r) &= \frac{1}{3} r^{-1/3}(3r^{1/3} - 5r^{1/6} + 2) \\ \Rightarrow \quad \rho'(1) &= \tfrac{1}{3}(3 - 5 + 2) = 0 \end{aligned} \tag{4.5.17}$$

Also by (4.5.15)

$$\begin{aligned} \rho(1) &= 1 - 2 + 1 = 0 \\ \therefore\ \rho'(1) &= \rho(1) = 0 \end{aligned}$$

Condition (iv)

From (4.5.17),

$$\begin{aligned} \rho''(z) &= \frac{1}{18} r^{-4/3}(5r^{1/6} - 4) \\ \Rightarrow \rho''(1) &= \frac{1}{18}(5 - 4) = \frac{1}{18} \end{aligned}$$

Using (4.5.16),

$$\begin{aligned} \sigma(1) &= \frac{4315 + 53994 - 2307 + 7948 - 4827 + 1578 - 221}{2177280} = \frac{1}{36} \\ 2!\sigma(1) &= 2\left(\frac{1}{36}\right) = \frac{1}{18} \\ \therefore\ \rho''(1) &= 2!\sigma(1) \end{aligned}$$

Thus by definition (4.1.6), the main method (3.3.60e) is consistent.

4.6.3 Zero Stability of the one step Method with Five Offstep Points

4.5.3.1 Zero stability of the Block Method (3.3.64)

Using (3.3.64) and the definitions in (3.2.12) in the limit as $h \to 0$ in (4.1.9), the first characteristic polynomial of the block method is obtained by (4.1.9) as

$$\rho(z) = \det\left[z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \right]$$

that is,

$$\rho(z) = \det \begin{bmatrix} z & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{bmatrix} = z^{11}(z-1) \qquad (4.5.18)$$

Equating (4.5.18) to zero and solving for z gives $z = 0$ or 1.

Clearly, no root has modulus greater than one and $|z| = 1$ is simple hence, the block method is zero stable in the limit as $h \to 0$ by definition (4.1.8).

4.5.3.2 Zero stability of the main method (3.3.60e)

Using equation (4.5.15), the roots of the first characteristic polynomial of the main method (3.3.60e) are obtained as $z = 0$ or 1. Since no root has modulus greater than one and $|z| = 1$ is simple, the method is zero stable by definition (4.1.7) and the zero stability of the block method (3.3.64)follows.

4.6.4 Convergence

If follows from Theorem 4.1.1 that the main method (3.3.60e) is convergent having established the consistency and zero stability of the method in sections (4.6.2) and (4.6.3)respectively.

4.6.5 Region of Absolute Stability of the One step Method with Five Offstep Points

The first and second characteristics polynomials of the main method (3.3.60e) have been given as (4.5.15) and (4.5.16) respectively. By the boundary Locus method, the boundary of the region of absolute stability is given by (4.1.14) where $z = e^{i\theta}$ as

$$\begin{aligned} \bar{h}(\theta) &= \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \\ \Rightarrow \bar{h}(\theta) &= \frac{2177280(e^{i\theta} - 2e^{i\frac{5}{6}\theta} + e^{i\frac{2}{3}\theta})}{4315e^{i\theta} + 53994e^{i\frac{5}{6}\theta} - 2307e^{i\frac{2}{3}\theta} + 7948e^{i\frac{1}{2}\theta} - 4827e^{i\frac{1}{3}\theta} + 1578e^{i\frac{1}{6}\theta} - 221} \end{aligned} \quad (4.5.19)$$

Writing (4.5.19) in terms of cosine and sine it reduces after some manipulations to the form

$$\bar{h}(\theta) = \frac{\begin{matrix} 4398105600\cos\frac{5}{6}\theta - 17862405120\cos\frac{2}{3}\theta + 41760230400\cos\frac{1}{2}\theta - 40747795200\cos\frac{1}{3}\theta \\ +243681177600\cos\frac{1}{6}\theta - 481178880\cos\theta - 230748134400 \end{matrix}}{\begin{matrix} 887868414\cos\frac{1}{3}\theta - 463460744\cos\frac{1}{2}\theta + 129767748\cos\frac{2}{3}\theta + 87506352\cos\frac{1}{6}\theta \\ -10247208\cos\frac{5}{6}\theta - 1907230\cos\theta + 3028303068 \end{matrix}} \quad (4.5.20)$$

Evaluating (4.5.20) at intervals of 30^0 gives the results in table 5.

Table 5. The boundaries of the region of absolute stability the One Step 5 offstep points method.

θ	0	30^0	60^0	90^0	120^0	150^0	180^0
$\bar{h}(\theta)$	0^0	-0.27	-1.10	-2.47	-4.36	-6.86	-9.87

From table 5, the interval of absolute stability is $(-9.87, 0)$. The region of absolute stability is shown figure 4.5 .

Remark

The locus is symmetric about x-axis as $x(-\theta) = x(\theta)$.

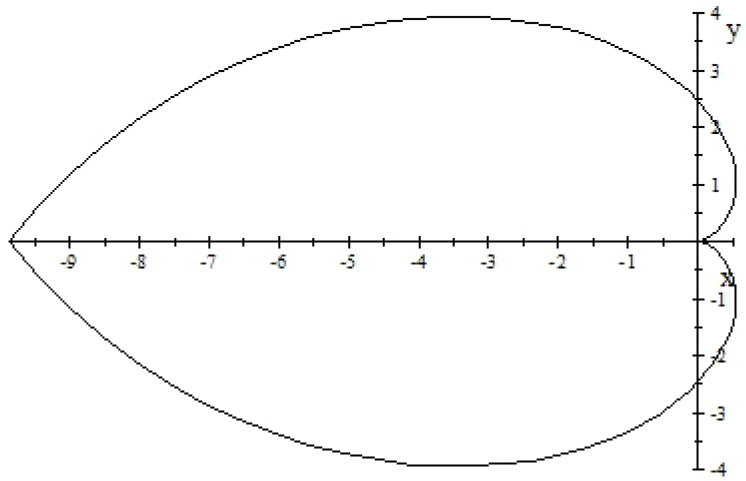

Figure 4.5: Region of Absolute Stability of the Continuous One-Step Method with Five Offstep Points

In what follows, a summary of the analysis of the methods is given as

Table 6. Summary of the analysis of the methods

Method	Order & Error Constant	Zero Stability	Consistency	Interval of Absolute Stability
1S1HM	P = 4, $c_{p+2} = -4.7291 \times 10^{-4}$	zero stable	consistent	-9.60, 0
1S2HM	P = 4, $c_{p+2} = -5.7158 \times 10^{-6}$	zero stable	consistent	-9.82, 0
1S3HM	P = 5, $c_{p+2} = -2.5431 \times 10^{-7}$	zero stable	consistent	-9.86, 0
1S4HM	P = 6, $c_{p+2} = -9.3545 \times 10^{-9}$	zero stable	consistent	-9.87, 0
1S5HM	P = 7, $c_{p+2} = -3.1173 \times 10^{-10}$	zero stable	consistent	-9.87, 0

Chapter 5

Implementation and Numerical Examples

5.1 Introduction

The implementation strategy for the methods is discussed in this chapter. Furthermore, the performance of the methods is tested on some numerical examples ranging from nonlinear, linear, to moderately stiff initial value problems of general second order ordinary differential equations. For each example, absolute error of the approximate solutions are computed and compared with results from existing methods particularly those proposed by Awoyemi (1999, 2001), Yahaya and Badmus (2009), Badmus and Yahaya (2009) and Jator (2007). The results from the methods are also discussed here.

5.2 Implementation

The strategy adopted for the implementation of the methods is such that all the discrete methods obtained from the continuous method as well as their derivatives, which have the same order of accuracy, with very low error constants for fixed h, are combined as simultaneous integrators. We proceed by explicitly obtaining initial conditions at x_{n+1} using values from the independent solutions of the simultaneous integrators over non-overlapping subintervals; $[0, x_1], \ldots, [x_{N-1}, x_N]$ (Yusuph & Onumanyi, 2005); to implement the respective methods proposed. The absolute errors calculated in the code are defined as

$$Erc = |yc - yex|$$

where yex is the exact solution, yc is the computed result and Erc is the absolute error.

All computations were carried out using FORTRAN codes in FORTRAN 95 language and executed on Windows XP operating system. The computer codes are simply written without the use of subroutines and requires no previous knowledge of programing before it can be used.

5.3 Numerical Examples

In order to study the efficiency of the developed methods, we present some numerical experiments with the following five problems. The Continuous Implicit Hybrid One Step Methods (CIHOSM): 1S1HM, 1S2HM, 1S3HM, 1S4HM and 1S5HM, were applied to solve the following test problems:

1. $xy'' - x + 3y' - \left(\frac{3}{x}\right) y,\ x_0 = 1,\ y(x_0) = 2,\ y'(x_0) = 10$
   Exact Solution: $y = 3x^3 - 2x + x^2(1 + x \ln x)$
   $h = \frac{0.1}{32}$
   Source: Awoyemi (1999)

2. $y'' - x(y')^2 = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$
   Exact solution: $y = 1 + \frac{1}{2} ln \left(\frac{2+x}{2-x}\right)$
   $h = \frac{0.1}{32}$
   Source: Awoyemi (2001)

3. $y'' = y', \;\; y(0) = 0, \;\; y'(0) = -1$

   Exact solution: $y(x) = 1 - \exp(x)$

   $h = 0.1$

   Source: Yahaya & Badmus (2009)

4. $y'' + 1001y' + 1000y = 0, \;\; y(0) = 1, \;\; y'(0) = -1$

   Exact solution: $y(x) = \exp(-x)$

   $h = 0.05$

   Source: Jator (2007)

5. $y'' + \left(\dfrac{6}{x}\right) y' + \left(\dfrac{4}{x^2}\right) y = 0, \;\; y(1) = 1, \;\; y'(1) = 1$

   Exact solution: $\dfrac{5}{3x} - \dfrac{3}{3x^4}$

   $h = \dfrac{0.1}{32}$

   Source: Badmus & Yahaya (2009)

The following notations are used in the tables

x	Point of Evaluation
Yex	Exact solution
1S1HM	one step, One offstep point method
1S2HM	one step, Two offstep points method
1S3HM	one step, Three offstep points method
1S4HM	one step, Four offstep points method
1S5HM	one step, Five offstep points method
Erc	Absolute error

The computed results for the five problems using the five methods proposed are presented in tables 7 to 16.

Table 7. Showing the exact solutions and the computed results from the proposed methods for Problem 1.

X	YEX	1S1HM	1S2HM	1S3HM	1S4HM	1S5HM
1.1	0.31298579E+01	0.31298580E+01	0.31298579E+01	0.31298579E+01	0.31298579E+01	0.31298579E+01
1.2	0.45390517E+01	0.45390516E+01	0.45390517E+01	0.45390517E+01	0.45390517E+01	0.45390517E+01
1.3	0.62574144E+01	0.62574142E+01	0.62574143E+01	0.62574143E+01	0.62574144E+01	0.62574144E+01
1.4	0.83152800E+01	0.83152797E+01	0.83152798E+01	0.83152799E+01	0.83152800E+01	0.83152799E+01
1.5	0.10743445E+02	0.10743445E+02	0.10743445E+02	0.10743445E+02	0.10743445E+02	0.10743445E+02
1.6	0.13573135E+02	0.13573135E+02	0.13573135E+02	0.13573135E+02	0.13573135E+02	0.13573135E+02
1.7	0.16835977E+02	0.16835976E+02	0.16835977E+02	0.16835977E+02	0.16835977E+02	0.16835977E+02
1.8	0.20563972E+02	0.20563971E+02	0.20563972E+02	0.20563972E+02	0.20563973E+02	0.20563972E+02
1.9	0.24789476E+02	0.24789475E+02	0.24789476E+02	0.24789476E+02	0.24789477E+02	0.24789476E+02
2.0	0.29545178E+02	0.29545176E+02	0.29545177E+02	0.29545178E+02	0.29545179E+02	0.29545178E+02

Table 8: Comparing the absolute errors in the new methods to errors in Awoyemi(1999) for Problem 1

X	Error in 1S1HM p=4, k=1	Error in 1S2HM p=4, k=1	Error in 1S3HM p=5, k=1	Error in 1S4HM p=6, k=1	Error in 1S5HM p=7, k=1	Error in Awoyemi(1999) p=4, k=2
1.1	0.12888106E-07	0.57475482E-08	0.24871327E-08	0.19055255E-08	0.14318067E-08	0.51125591E-07
1.2	0.54913644E-07	0.24458675E-07	0.10363642E-07	0.80201747E-08	0.60836944E-08	–
1.3	0.13065430E-06	0.58169761E-07	0.24154704E-07	0.18919443E-07	0.14459401E-07	0.52074859E-06
1.4	0.24455343E-06	0.10885790E-06	0.44309187E-07	0.35151302E-07	0.27047847E-07	–
1.5	0.40094191E-06	0.17845020E-06	0.71207941E-07	0.57235781E-07	0.44325446E-07	0.15996662E-05
1.6	0.60406071E-06	0.26883387E-06	0.10517184E-06	0.85669157E-07	0.66758535E-07	–
1.7	0.85807786E-06	0.38186365E-06	0.14646803E-06	0.12092716E-06	0.94805369E-07	0.34250638E-05
1.8	0.11671012E-05	0.51936759E-06	0.19531497E-06	0.16346750E-06	0.12891749E-06	–
1.9	0.15351888E-05	0.68315154E-06	0.25188674E-06	0.21373182E-06	0.16954080E-06	0.61293258E-05
2.0	0.19663562E-05	0.87500259E-06	0.31631626E-06	0.27214757E-06	0.21711659E-06	0.78514782E-05

Table 9. Showing the exact solutions and the computed results from the proposed methods for Problem 2

X	YEX	1S1HM	1S2HM	1S3HM	1S4HM	1S5HM
0.1	0.10500417E+01	0.10500417E+01	0.10500417E+01	0.10500417E+01	0.10500417E+01	0.10500417E+01
0.2	0.11003353E+01	0.11003353E+01	0.11003353E+01	0.11003353E+01	0.11003354E+01	0.11003354E+01
0.3	0.11511404E+01	0.11511404E+01	0.11511404E+01	0.11511404E+01	0.11511404E+01	0.11511404E+01
0.4	0.12027326E+01	0.12027326E+01	0.12027326E+01	0.12027326E+01	0.12027326E+01	0.12027326E+01
0.5	0.12554128E+01	0.12554128E+01	0.12554128E+01	0.12554128E+01	0.12554128E+01	0.12554128E+01
0.6	0.13095196E+01	0.13095196E+01	0.13095196E+01	0.13095196E+01	0.13095196E+01	0.13095196E+01
0.7	0.13654438E+01	0.13654437E+01	0.13654438E+01	0.13654438E+01	0.13654438E+01	0.13654438E+01
0.8	0.14236489E+01	0.14236489E+01	0.14236489E+01	0.14236489E+01	0.14236489E+01	0.14236489E+01
0.9	0.14847003E+01	0.14847002E+01	0.14847003E+01	0.14847003E+01	0.14847003E+01	0.14847003E+01
1.0	0.15493062E+01	0.15493060E+01	0.15493061E+01	0.15493061E+01	0.15493062E+01	0.15493062E+01

Table 10. Comparing the absolute errors in the new methods to errors in Awoyemi(2001) for Problem 2.

X	Error in 1S1HM p=4, k=1	Error in 1S2HM p=4, k=1	Error in 1S3HM p=5, k=1	Error in 1S4HM p=6, k=1	Error in 1S5HM p=7, k=1	Error in Awoyemi(2001) p=6, k=4
0.1	0.49827253E-10	0.22413627E-10	0.12381429E-10	0.76769702E-11	0.69253492E-11	0.26075253E-09
0.2	0.41043058E-09	0.18330515E-09	0.10142642E-09	0.62873262E-10	0.56260108E-10	0.19816704E-08
0.3	0.14285815E-08	0.63686190E-09	0.35245495E-09	0.21852276E-09	0.19460211E-09	0.65074122E-08
0.4	0.35242687E-08	0.15698642E-08	0.86895491E-09	0.53898974E-09	0.47769566E-09	0.15592381E-07
0.5	0.72435324E-08	0.32251910E-08	0.17857029E-08	0.11083281E-08	0.97695985E-09	0.31504477E-07
0.6	0.13335597E-07	0.59361429E-08	0.32879226E-08	0.20423685E-08	0.17889177E-08	0.56374577E-07
0.7	0.22872871E-07	0.10179837E-07	0.56411280E-08	0.35075920E-08	0.30497800E-08	0.96164046E-07
0.8	0.37447019E-07	0.16664477E-07	0.92399235E-08	0.57519813E-08	0.49590416E-08	0.15686801E-06
0.9	0.59503708E-07	0.26478362E-07	0.14691385E-07	0.91578443E-08	0.78195337E-08	0.24869769E-06
1.0	0.92940412E-07	0.41356017E-07	0.22964069E-07	0.14336313E-07	0.12108436E-07	0.38798389E-06

Table 11. Showing the exact solutions and the computed results from the proposed methods for Problem 3

X	YEX	1S1HM	1S2HM	1S3HM	1S4HM	1S5HM
0.1	-.10517092E+00	-.10517083E+00	-.10517084E+00	-.10517084E+00	-.10517084E+00	-.10517083E+00
0.2	-.22140276E+00	-.22140813E+00	-.22139976E+00	-.22140100E+00	-.22140159E+00	-.22140192E+00
0.3	-.34985881E+00	-.34986506E+00	-.34985027E+00	-.34985389E+00	-.34985560E+00	-.34985654E+00
0.4	-.49182471E+00	-.49182622E+00	-.49180748E+00	-.49181483E+00	-.49181830E+00	-.49182020E+00
0.5	-.64872128E+00	-.64871127E+00	-.64869162E+00	-.64870433E+00	-.64871032E+00	-.64871360E+00
0.6	-.82211882E+00	-.82208911E+00	-.82207227E+00	-.82209227E+00	-.82210167E+00	-.82210682E+00
0.7	-.10137527E+01	-.10136936E+01	-.10136841E+01	-.10137136E+01	-.10137275E+01	-.10137351E+01
0.8	-.12255410E+01	-.12254407E+01	-.12254440E+01	-.12254858E+01	-.12255054E+01	-.12255161E+01
0.9	-.14596031E+01	-.14594482E+01	-.14594708E+01	-.14595279E+01	-.14595546E+01	-.14595693E+01
1.0	-.17182819E+01	-.17180560E+01	-.17181056E+01	-.17181817E+01	-.17182174E+01	-.17182369E+01

Table 12. Comparing the absolute errors in the new methods to errors in Yahaya & Badmus(2009) for Problem 3.

X	Error in 1S1HM p=4, k=1	Error in 1S2HM p=4, k=1	Error in 1S3HM p=5, k=1	Error in 1S4HM p=6, k=1	Error in 1S5HM p=7, k=1	Error in Yahaya & Badmus(2009) p=4, k=2
0.1	0.84742321E-07	0.81573327E-07	0.84742321E-07	0.84736252E-07	0.84742330E-07	0.87931600E-04
0.2	0.53721850E-05	0.29974140E-05	0.17614744E-05	0.11719652E-05	0.84643034E-06	0.32671800E-03
0.3	0.62472499E-05	0.85399753E-05	0.49277699E-05	0.32170472E-05	0.22755542E-05	0.22155640E-02
0.4	0.15165650E-05	0.17229468E-04	0.98783854E-05	0.64094269E-05	0.45040757E-05	0.48570930E-02
0.5	0.10008420E-04	0.29666467E-04	0.16953599E-04	0.10967802E-04	0.76843308E-05	0.90977340E-02
0.6	0.29704225E-04	0.46543059E-04	0.26545528E-04	0.17144180E-04	0.11991856E-04	0.14391394E-01
0.7	0.59161730E-04	0.68655449E-04	0.39105262E-04	0.25228466E-04	0.17628582E-04	0.21437918E-01
0.8	0.10021603E-03	0.96918204E-04	0.55150938E-04	0.35553649E-04	0.24826447E-04	0.29898724E-01
0.9	0.15498023E-03	0.13238034E-03	0.75276848E-04	0.48501651E-04	0.33851471E-04	0.40300719E-01
1.0	0.22588355E-03	0.17624351E-03	0.10016373E-03	0.64509947E-04	0.45008361E-04	0.52552130E-01

Table 13. Showing the exact solutions and the computed results from the proposed methods for Problem 4

X	YEX	1S1HM	1S2HM	1S3HM	1S4HM	1S5HM
0.1	0.90483742E+00	0.90483742E+00	0.90483742E+00	0.90483742E+00	0.90483742E+00	0.90483742E+00
0.2	0.81873075E+00	0.81873075E+00	0.81873075E+00	0.81873075E+00	0.81873075E+00	0.81873075E+00
0.3	0.74081822E+00	0.74081822E+00	0.74081822E+00	0.74081822E+00	0.74081822E+00	0.74081822E+00
0.4	0.67032004E+00	0.67032004E+00	0.67032004E+00	0.67032004E+00	0.67032004E+00	0.67032004E+00
0.5	0.60653066E+00	0.60653065E+00	0.60653066E+00	0.60653066E+00	0.60653066E+00	0.60653066E+00
0.6	0.54881163E+00	0.54881163E+00	0.54881163E+00	0.54881163E+00	0.54881163E+00	0.54881163E+00
0.7	0.49658530E+00	0.49658530E+00	0.49658530E+00	0.49658530E+00	0.49658530E+00	0.49658530E+00
0.8	0.44932896E+00	0.44932896E+00	0.44932896E+00	0.44932896E+00	0.44932896E+00	0.44932896E+00
0.9	0.40656965E+00	0.40656965E+00	0.40656965E+00	0.40656965E+00	0.40656965E+00	0.40656965E+00
1.0	0.36787944E+00	0.36787944E+00	0.36787944E+00	0.36787944E+00	0.36787944E+00	0.36787944E+00

Table 14. Comparing the absolute errors in the new methods to errors in Jator(2007) for Problem 4.

X	Error in 1S1HM p=4, k=1	Error in 1S2HM p=4, k=1	Error in 1S3HM p=5, k=1	Error in 1S4HM p=6, k=1	Error in 1S5HM p=7, k=1	Error in Jator(2007) p=6, k=5
0.1	0.10886170E-09	0.21485147E-10	0.13039125E-10	0.24604763E-11	0.23759883E-11	0.698677E-11
0.2	0.20752355E-09	0.42134740E-10	0.23166580E-10	0.67051920E-11	0.65671912E-11	0.100275E-11
0.3	0.28642155E-09	0.58659744E-10	0.31248448E-10	0.10121015E-10	0.99376063E-11	0.785878E-11
0.4	0.34842440E-09	0.71659456E-10	0.37582271E-10	0.12827850E-10	0.12605583E-10	0.104778E-10
0.5	0.39603265E-09	0.81653351E-10	0.42427728E-10	0.14927615E-10	0.14674928E-10	0.632212E-10
0.6	0.43142434E-09	0.89093510E-10	0.46010196E-10	0.16511903E-10	0.16242563E-10	0.100508E-10
0.7	0.45649384E-09	0.94378005E-10	0.48526738E-10	0.17659374E-10	0.17379265E-10	0.936336E-11
0.8	0.47288495E-09	0.97850172E-10	0.50147830E-10	0.18438973E-10	0.18152535E-10	0.264766E-11
0.9	0.48202237E-09	0.99806885E-10	0.51020632E-10	0.18909985E-10	0.18621049E-10	0.106793E-10
1.0	0.48513832E-09	0.10050460E-09	0.51273152E-10	0.19124535E-10	0.18836210E-10	0.232731E-10

Table 15. Showing the exact solutions and the computed results from the proposed methods for Problem 5.

X	YEX	1S1HM	1S2HM	1S3HM	1S4HM	1S5HM
1.0031	0.10030765E+01	0.10030765E+01	0.10030765E+01	0.10030765E+01	0.10030765E+01	0.10030765E+01
1.0063	0.10060575E+01	0.10060575E+01	0.10060575E+01	0.10060575E+01	0.10060575E+01	0.10060575E+01
1.0094	0.10089450E+01	0.10089450E+01	0.10089450E+01	0.10089450E+01	0.10089450E+01	0.10089450E+01
1.0125	0.10117410E+01	0.10117410E+01	0.10117410E+01	0.10117410E+01	0.10117410E+01	0.10117410E+01
1.0156	0.10144475E+01	0.10144476E+01	0.10144476E+01	0.10144475E+01	0.10144475E+01	0.10144475E+01
1.0188	0.10170665E+01	0.10170665E+01	0.10170665E+01	0.10170665E+01	0.10170665E+01	0.10170665E+01
1.0219	0.10195998E+01	0.10195998E+01	0.10195998E+01	0.10195998E+01	0.10195998E+01	0.10195998E+01
1.0250	0.10220492E+01	0.10220492E+01	0.10220492E+01	0.10220492E+01	0.10220492E+01	0.10220492E+01
1.0281	0.10244165E+01	0.10244166E+01	0.10244166E+01	0.10244165E+01	0.10244165E+01	0.10244165E+01
1.0313	0.10267036E+01	0.10267037E+01	0.10267036E+01	0.10267036E+01	0.10267036E+01	0.10267036E+01

Table 16. Comparing the absolute errors in the new methods to errors in Badmus & Yahaya (2009) for Problem 5.

X	Error in 1S1HM p=4, k=1	Error in 1S2HM p=4, k=1	Error in 1S3HM p=5, k=1	Error in 1S4HM p=6, k=1	Error in 1S5HM p=7, k=1	Error in Badmus & Yahaya(2009) p=4, k=2
1.0031	0.81807894E-11	0.77022833E-11	0.77009510E-11	0.77586826E-11	0.76998408E-11	0.38354E-04
1.0063	0.27785485E-08	0.12658126E-08	0.71779915E-09	0.45977178E-09	0.41327297E-09	0.75004E-04
1.0094	0.74754114E-08	0.33850827E-08	0.19176578E-08	0.12299799E-08	0.10432657E-08	0.10592E-03
1.0125	0.13997774E-07	0.63126238E-08	0.35708470E-08	0.22894449E-08	0.18729474E-08	0.13548E-03
1.0156	0.22264084E-07	0.10012317E-07	0.56571030E-08	0.36252337E-08	0.28933289E-08	0.15557E-03
1.0188	0.32195759E-07	0.14449356E-07	0.81569016E-08	0.52248852E-08	0.40957480E-08	0.18637E-03
1.0219	0.43717076E-07	0.19590206E-07	0.11051428E-07	0.70763917E-08	0.54718601E-08	0.19606E-03
1.0250	0.56755078E-07	0.25402552E-07	0.14322554E-07	0.91681831E-08	0.70136241E-08	0.22104E-03
1.0281	0.71239469E-07	0.31855260E-07	0.17952815E-07	0.11489112E-07	0.87132936E-08	0.20563E-03
1.0313	0.87102518E-07	0.38918333E-07	0.21925381E-07	0.14028440E-07	0.99304354E-08	0.27791E-03

5.4 Discussion of the Results

Computer programs written for the implementation of the five Continuous Implicit Hybrid One Step Methods (CIHOSM) developed namely: 1S1HM, 1S2HM, 1S3HM, 1S4HM and 1S5HM; were tested respectively on five numerical examples which are, respectively, nonlinear, linear and stiff initial value problems of general second order ordinary differential equations in the last section.

The approximate solutions obtained from these experiments elucidated the efficiency of the computed programs.

It is observed from the tables that the results obtained from the methods converged faster when the number of offstep points were increased. This validates the consistency and zero stability of the methods and agrees with the fact that, as the step size h decreases, the methods get more accurate as demonstrated in Table 5. Even though there are some deviations in what is obtained in Table 12, where the results obtained at $x = 0.3$ to 0.7 by 1S1HM are better than those from 1S2HM for the moderately stiff problem 3.

Generally, the performance of our methods as noticed in tables 7 to 16, are superior to those from methods implemented on the predictor-corrector codes by Awoyemi (1999 and 2001) and the block methods proposed by Yahaya and Badmus (2009) and Badmus and Yahaya (2009) for the same step sizes, even though their method had higher step numbers.

However, even though the multiple finite difference method of Jator (2007) seemed to have produced a better result at most of the points of evaluation, it should be noticed that the method had step number $k = 5$ against our methods with step number $k = 1$. Indeed, our methods compared favourably with Jator (2007) results given the obvious differences in their designs.

It should be noticed that, as the offstep points are decreasing, the step size becomes smaller and the methods become more accurate for the five problems used to test the accuracy of the methods just as small steps in the finite elements methods increase the accuracy of the problems solved.

Also, beyond the reduction in step number which means lesser function evaluations per iteration, this approach produced higher order discrete methods which give very low error terms and wider intervals of absolute stability.

Chapter 6

Summary and Conclusion

6.1 Introduction

In this chapter, a general conclusion on the research work is made. Recommendations on the proposed class of methods is made while areas of further research are suggested.

6.2 Summary and Conclusion

A class of hybrid collocation methods for the direct solution of initial value problems of general second-order ordinary differential equations have been developed in this research. The collocation technique yielded very consistent and zero stable implicit hybrid one step methods with continuous coefficients. The methods are implemented without the need for the development of predictors nor requiring any other method to generate starting values. Furthermore, the inclusion of offstep points allowed the adoption of linear multistep procedure, circumvent the 'zero-stability barrier', upgraded the order of accuracy of the methods and to obtain very low error constants.

In particular, the performance of the methods improved as the number of offstep points increased. Adequate stability intervals are obtained for both non-stiff and stiff problems. Results from the numerical solutions of non-linear, linear and moderately stiff IVP show that this class of methods are superior to the predictor-corrector method proposed by Awoyemi (1999, 2001), the multiple finite difference methods proposed by Jator (2007) and the hybrid block method proposed by Yahaya and Badmus (2009) and Badmus and Yahaya (2009).

All computations were carried out by computer programs written in FORTRAN 95 language, compiled and executed using PLATO FORTRAN 95 on the Windows XP operating system. The computer programs do not contain subroutines which means lesser computing time.

6.3 Open Problems

The class of continuous Implicit one step hybrid methods proposed in this thesis is recommended for the direct solution of initial value problems of general second order ordinary differential equations; of the nonlinear, linear and stiff types for small h, as demonstrated by the results of this research in terms of efficiency and accuracy and ease of implementation.

However, further research could extend the methods to the direct solution of higher order general ODEs in view of the advantages of these methods.

The plausibility of using other basis functions in place of the power series polynomial used in this work is also suggested for further research in this direction.

Systems of special and general higher order ODEs can be considered using the new methods proposed in this work.

6.4 Contribution to Knowledge

This research has led to the following contributions:

(i) a new class of continuous implicit hybrid one step methods for the direct solution of general second-order IVPs of ODE has been developed;

(ii) a new hybrid block formula has been defined; and

(iii) very accurate and highly efficient computer codes have been written for the implementation of the new methods

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Appendices