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He's Polynomials for Analytical Solutions of the Black-Scholes Pricing Model for Stock Option Valuation

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***Abstract*— The Black-Scholes model is one of the most famous and useful models for option valuation as regards option pricing theory. In this paper, we propose a semi-analytical method referred to as He's polynomials for solving the classical Black-Scholes pricing model with stock as the underlying asset. The proposed method gives the exact solution of the solved problem in a very simple and quick manner even with less computational work while still maintaining high level of accuracy. Hence, we recommend an extension and adoption of this method for solving problems arising in other areas of financial engineering, finance, and applied sciences.**

***Index Terms*— Analytical solutions; He's polynomials; Black-Scholes model, stock option valuation**

I. INTRODUCTION

Financial engineering appears to be among the most and famous areas of applied mathematics from both theoretical, industrial and academic point of view. In the theory of finance, and options valuation; options are referred to as financial contracts furnishing its holder the right but not an obligation to be involved in a future transaction on some underlying securities [1]. Therefore, an option having stock as its underlying asset is referred to as stock option. One of the most fundamental aspects of financial mathematics is hinged on modelling of evolution of financial processes such as interest rates, rates of exchange, stock prices, and so on [2-4].

In order to control the risk induced by the movements of stock prices, options can be used for hedging assets and portfolios. With regard to theory of option pricing and valuation, Black and Scholes in 1973 [5] proposed a classical formula for the prices of financial options. This is popularly referred to as Black-Scholes equation, which has been the hallmark of financial derivatives. The Black-Scholes model is a linear PDE based on some assumptions whose modification and relaxation have also led to various forms with nonlinear cases inclusive [6,7].

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II. THE BLACK-SCHOLES MODEL

In financial engineering, the Black-Scholes model is a parabolic differential equation whose solution is used in describing the value of European option [8]. In these recent days, the Black-Scholes model has been extended for valuation of options other than European options.

In what follows, we will consider the classical celebrated Black–Scholes option pricing model:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}-rf=0 \qquad (1)$$

where $f=f(S,\tau)$ is the value of the contingent claim S, at time, τ
$0\le\tau\le t,\ (S,\tau)\in R^+\times(0,T),\ f\in C^{2,1}\left[R\times[0,T]\right]$
with a payoff function $p_f(S,t)$, and expiration price, E such that:

$$p_f(S,t)=\begin{cases}\max(S-E,0), \text{ for European call option}\\ \max(E-S,0), \text{ for European put option}\end{cases} \qquad (2)$$

where $\max(S_*,0)$ indicates the large value between S_* and 0, σ denotes the volatility of the underlying asset $S=S(t)$, r is the risk-free interest rate, T is the time at maturity. Note, the randomness of $f=f(S,t)$ will be totally correlated to that of $S=S(t)$.

Recently, many researchers have considered some numerical, and analytical methods for solving the Black–Scholes model, and other option valuation models. These include Adomian decomposition method (ADM), variational iteration method (VIM), Differential transformation method (DTM), projected DTM (PDTM), homotopy perturbation method (HPM), [9-15].

The He's polynomials was introduced by Ghorbani *et al.* [16, 17] where nonlinear terms were split into a series of polynomials which are calculated from HPM. It is remarked that He's polynomials are compatible with Adomian's polynomials, yet it is showed that the He's polynomials are easier to compute, and are very much user friendly [18-20].

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Our goal is therefore, to provide analytical solutions to the Black–Scholes option pricing model of the form in (1) using a proposed semi-analytical technique referred to as He's polynomials method.

It is worth mentioning that the He's polynomials method is an alternative semi-analytical method, even without giving up accuracy.

III. *OVERVIEW OF THE HE'S POLYNOMIAL METHOD*

In general form, we consider the equation:

$$\Im(\psi) = 0 \tag{3}$$

where $\Im$ is an integral or a differential operator. Let $H(\psi, p)$ be a convex homotopy defined by:

$$H(\psi, p) = p\Im(\psi) + (1-p)G(\psi) \tag{4}$$

where $G(\psi)$ is a functional operator with ψ_0 as a known solution. Thus, we have:

$$H(\psi, 0) = G(\psi) \text{ and } H(\psi, 1) = \Im(\psi) \tag{5}$$

whenever $H(\psi, p) = 0$ is satisfied, and $p \in (0,1]$ is an embedded parameter. In HPM [18,19], p is used as an expanding parameter to obtain:

$$\psi = \sum_{j=0}^{\infty} p^j \psi_j = \psi_0 + p\psi_1 + p^2\psi_2 + \cdots \tag{6}$$

From (6) the solution is obtained as $p \to 1$. The convergence of (9) as $p \to 1$ has been considered in [20].

The method considers $N(\psi)$ (the nonlinear term) as:

$$N(\psi) = \sum_{j=0}^{\infty} p^j H_j = H_o + p^1 H_1 + p^2 H_2 + p^3 H_3 + \cdots \tag{7}$$

where H_k's are the so-called He's polynomials, which can be computed using:

$$H_k = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left(N\left(\sum_{j=0}^{k} p^j \psi_j \right) \right)_{p=0}, \ n \geq 0 \tag{8}$$

where $H_k = H_k(\psi_0, \psi_1, \psi_2, \psi_3, \cdots, \psi_k)$

IV. *THE HE'S POLYNOMIALS AND THE PRICING MODEL*

In this subsection, the He's Polynomials approach will be applied to the model equation as follows:

Problem 1: Consider the following linear Black-Scholes equation:

$$\frac{\partial w}{\partial t} + x^2 \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} x \frac{\partial w}{\partial x} - w = 0 \tag{9}$$

subject to:

$$w(x,0) = \max(x^3, 0) = \begin{cases} x^3, \text{ for } x > 0 \\ 0, \text{ for } x \leq 0 \end{cases} \tag{10}$$

Procedure w.r.t Problem 1:

We re-write (9) as (11) and in an integral form, we reformulate (11) as (12) below:

$$\frac{\partial w}{\partial t} = -x^2 \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} x \frac{\partial w}{\partial x} + w \tag{11}$$

$$w(x,t) = \max(x^3, 0) - p\int_0^t \left\{ x^2 \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} x \frac{\partial w}{\partial x} - w \right\} dt \tag{12}$$

Applying the convex homotopy method to (12) gives:

$$\sum_{n=0}^{\infty} p^n w_n = \max(x^3, 0)$$
$$-p\int_0^t \left\{ x^2 \sum_{n=0}^{\infty} p^n \frac{\partial^2 w_n}{\partial x^2} + \frac{x}{2} \sum_{n=0}^{\infty} p^n \frac{\partial w_n}{\partial x} - \sum_{n=0}^{\infty} p^n w_n \right\} dt \tag{13}$$

$$\Rightarrow w_0 + pw_1 + p^2 w_2 + \cdots = \max(x^3, 0)$$
$$-p\int_0^t \left\{ x^2 \left(\frac{\partial^2 w_0}{\partial x^2} + p\frac{\partial^2 w_1}{\partial x^2} + p^2 \frac{\partial^2 w_2}{\partial x^2} + \cdots \right) \right.$$
$$+ \frac{x}{2}\left(\frac{\partial w_0}{\partial x} + p\frac{\partial w_1}{\partial x} + p^2 \frac{\partial w_2}{\partial x} + \cdots \right)$$
$$\left. + -\left(w_0 + pw_1 + p^2 w_2 + \cdots \right) \right\} dt \tag{14}$$

We compare the coefficients of the equal powers of p in the following ways:

$$p^{(0)} : w_0 = \max(x^3, 0)$$

$$p^{(1)} : w_1 = -\int_0^t \left(x^2 \frac{\partial^2 w_0}{\partial x^2} + \frac{x}{2}\frac{\partial w_0}{\partial x} - w_0 \right) dt$$

$$p^{(2)} : w_2 = -\int_0^t \left(x^2 \frac{\partial^2 w_1}{\partial x^2} + \frac{x}{2}\frac{\partial w_1}{\partial x} - w_1 \right) dt$$

$$p^{(3)} : w_3 = -\int_0^t \left(x^2 \frac{\partial^2 w_2}{\partial x^2} + \frac{x}{2}\frac{\partial w_2}{\partial x} - w_2 \right) dt$$

$$\vdots$$

$$p^{(m)} : w_m = -\int_0^t \left(x^2 \frac{\partial^2 w_{m-1}}{\partial x^2} + \frac{x}{2}\frac{\partial w_{m-1}}{\partial x} - w_{m-1} \right) dt$$

Therefore, simplifying $p^{(1)}$, $p^{(2)}$, $p^{(3)}$, and so on using $w_0 = \max(x^3, 0)$, for $x \geq 0$ gives:

$$w_0 = x^3, \ w_1 = -\frac{13}{2}x^3 t, \ w_2 = \frac{169}{8}x^3 t^2, \ w_3 = \frac{-2197}{48}x^3 t^3,$$
$$w_4 = \frac{28561}{384}x^3 t^4, \ w_5 = \frac{-371293}{3840}x^3 t^5, \cdots$$

$$\therefore \quad w(x,t) = w_0 + w_1 + w_2 + w_3 + w_4 + w_5 + \cdots$$
$$= x^3 - \frac{13}{2}x^3 t + \frac{169}{8}x^3 t^2 - \frac{2197}{48}x^3 t^3$$
$$+ \frac{28561}{384}x^3 t^4 - \frac{371293}{3840}x^3 t^5 + \cdots$$
$$= x^3 \left(1 + (-6.5t) + \frac{(-6.5t)^2}{2!} + \frac{(-6.5t)^3}{3!} \right.$$

$$+\frac{(-6.5t)^4}{4!}+\frac{(-6.5t)^5}{5!}+\cdots\Bigg)$$
$$\cong x^3 e^{-6.5t} \tag{15}$$

Note: Figure 1 and Figure 2 below represent the graphs of the exact solution and the He's polynomials solution (including terms up to power 5) respectively:

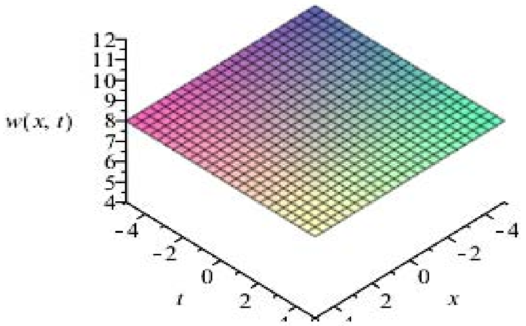

Figure1: for the exact solution

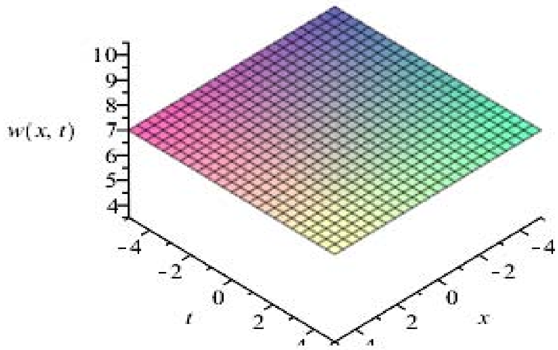

Figure2: for the approximate solution

V. Concluding Remarks

In this paper, we have succeeded in providing an accurate and exact solutions to the classical linear Black-Scholes model for stock option valuation. This is done via the application of He's polynomials as the proposed technique. This technique is very much efficient and reliable as it gives the exact solution of the solved problem in a very simple and quick manner even with less computational work while still maintaining high level of accuracy. Hence, we recommend an extension and adoption of this method for solving problems arising in other areas of applied sciences: financial engineering and finance. We used *Maple* 18 for all numerical computations, and graphics done in this study.

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AUTHOR CONTRIBUTIONS

The concerned authors: SOE, OOU, and EAO contributed positively to this work. They all read and approved the final manuscript for publication.

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