Finite Difference Dynamic Analysis of Railway Bridges Supported by Pasternak Foundation under Uniform Partially Distributed Moving Railway Vehicle

M. C. Agarana and J. A. Gbadeyan

Abstract— **Rail transport has experienced great advances in recent times, characterised by increasing high speed and weights of railway vehicles. The vibration and dynamic stress being subjected to by the transport structures, such as road or railway bridges, have increased due to these factors. In this paper, the dynamic response of railway bridges, modelled as an elastic rectangular plate, continuously supported by Pasternak foundation and traversed by moving railway vehicle is investigated. Finite difference method is used to transform the set of coupled partial differential equations to a set of algebraic equations. The desired solutions are obtained with the aid of computer programs developed in conjunction with MATLAB. It is observed that the deflection of the railway bridge decreases as the foundation moduli increase. The rotatory inertia and shear deformation have significant effect on the deflection of the railway bridge under a moving railway vehicle (modelled as partially distributed moving load).**

Index Terms— **Dynamic response, finite difference method, Pasternak foundation, railway bridges**

I. Introduction

The moving load problem is a fundamental problem in several fields of Applied Mathematics, Mechanical Engineering, Applied Physics and Railway Engineering. The importance of this problems also manifested in numerous applications in the area of railway transportation. Rails and bridges are examples of structural elements to be designed to support moving masses [1]. Also recently an attempt has been made to analyse the dynamic response of a Mindlin Elastic plate under the influence of moving load, without considering the influence of rotatory inertia and shear deformation on the plate [2].

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While one of the works of Gbadeyan and Dada [3] also considered the dynamic response of elastic rectangular Mindlin plates under uniform partially distributed moving mass[15]. For practical application, it is useful to consider plates supported by an elastic foundation. For instance, an analysis involving such foundation can be used to determine the behaviour of bridges traversed by rail vehicle. Furthermore, structural members, especially, plates resting on elastic foundation have wide applications in modern engineering practices such as railway bridges, highway pavements and continuously supported pipelines [1,6,10].

In the present work, the model suggested in reference [2,3] is extended to include the effect of foundation reaction on the vibration of railway bridge (modelled as Mindlin plate)[1]. The foundation reaction is modelled as Pasternak type [10]. An attempt is therefore made in this paper to carry out a dynamic analysis of reactions of Railway Bridge, as an elastic structure, on elastic foundation under the influence of an external moving load - railway vehicle.

II. Problem Definition

A railway bridge, modelled as a rectangular plate, with a moving railway vehicle (moving load) and different boundary conditions is considered. The load is relatively large, that is, its inertia cannot be neglected, and is moving along the mid-space on the surface of the bridge, supported by a Pasternak foundation, as shown in figure 1.[1]

A. Assumptions

(i). The railway bridge is of constant cross – section, (ii.) the moving railway vehicle moves with a constant speed, (iii). The moving railway vehicle is guided in such a way that it keeps contact with the plate throughout the motion, (iv). The railway bridge is continuously supported by a Pasternak foundation, (v). The moving railway vehicle is partially distributed, (vi). The railway bridge ,as a plate, is elastic, (vii). No damping in the system, (viii). Uniform gravitational field and (ix). Constant mass (M_L) of the railway vehicle on the railway bridge.

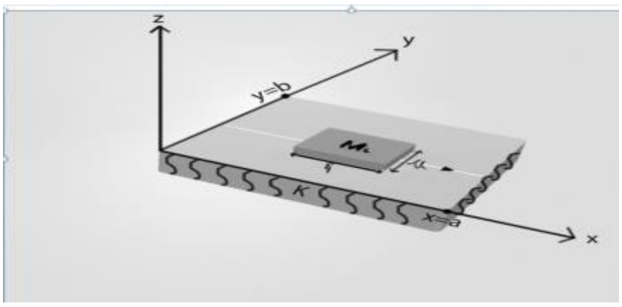

Figure I. A moving railway vehicle on the railway bridge supported by Pasternak foundation

B. Initial Conditions

W (x, y, o) = 0 = $\frac{\partial W}{\partial T}$ (x, y, 0)

C. Boundary Conditions

W (x, y, t) = M_X (x, y, t) = ψ_Y (x, y, t) = 0, for x = 0 and x = a
W (x, y, t) = M_Y (x, y, t) = ψ_X (x, y, t) = 0, for y = 0 and y = b

Where M_X and M_y are bending moments in the x – and y – directions respectively, ψ_X (x, y, t) and ψ_y (x, y, t) are local rotations in the x – and y – directions respectively. $W\ (x, y, t)$ is the traverse displacement of the plate at time t.

III. Problem Solution

The set of dynamic equilibrium equations which govern the behaviour of Mindlin plate supported by Pasternak foundation and traversed by a partially distributed moving load may be written as [1,3];

$$Q_x - \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = \frac{\rho h^3}{12}\frac{\partial^2 \varphi_x}{\partial T^2} + \frac{\rho_L h_1^3}{12} \left[\frac{\partial^2 \varphi_x}{\partial T^2} + \mathrm{U}\frac{\partial^2 \varphi_x}{\partial_x \partial T^2} + \frac{\mathrm{U}}{D(\upsilon^2-1)}\left\{\frac{\partial M_x}{\partial T} + \mathrm{U}\frac{\partial M_x}{\partial x}\right\} - \frac{\mathrm{U}\upsilon}{D(\upsilon^2-1)}\left\{\frac{\partial M_x}{\partial T} + \mathrm{U}\frac{\partial M_x}{\partial x}\right\}\right] B \tag{1}$$

$$Qy - \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} = \frac{\rho h^3}{12}\frac{\partial^2 \varphi_x}{\partial T^2} + \frac{\rho_L h_1^3}{12} \left[\frac{\partial^2 \varphi_x}{\partial T^2} + \mathrm{U}\frac{\partial^2 \varphi_x}{\partial_x \partial T^2} + \frac{\mathrm{U}}{D(\upsilon^2-1)}\left\{\frac{\partial M_x}{\partial T} + \mathrm{U}\frac{\partial M_x}{\partial x}\right\} - \frac{\mathrm{U}\upsilon}{D(\upsilon^2-1)}\left\{\frac{\partial M_x}{\partial T} + \mathrm{U}\frac{\partial M_x}{\partial x}\right\}\right] B \tag{2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + kW + (M_f - \boldsymbol{\rho h})\frac{\partial D_T}{\partial T} + \frac{M_L}{A}\left[g + \frac{\partial D_T}{\partial T} + U\frac{\partial D_T}{\partial T} + G\ \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y}\right) + \left\{\frac{\partial \psi_x}{\partial T} + \frac{U}{D(\upsilon^2-1)}M_x - \frac{U\upsilon}{D(\upsilon^2-1)}M_y\right\} - \frac{U}{\propto Gh}\left\{\frac{\partial Q_x}{\partial T} + U\frac{\partial Q_x}{\partial x}\right\}\right]$$

$$B = \boldsymbol{\rho} h\frac{\partial^2 \varphi_x}{\partial T^2} \tag{3}$$

where ψ_x and ψ_y are local rotations in the x – and y – directions respectively. M_x and M_y bending moments in the x- and y- directions respectively, M_{xy} is the twitting moments, Q_x and Q_y are the traversed Shearing forces in x – and y – directions respectively, h and h_1 are thickness of the plate and load respectively, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_L$ are the densities of the plate and the load per unit volume respectively W(x,y,T) is the traverse displacement of the plate at time T, P(x,y,T) is the applied dynamic load (force) and the last terms in equation (1) and (2) account for inertia effects of the load in x – and y – directions respectively. It is the velocity of a load (M_L) of rectangular dimensions E by U with one of its lines of symmetry moving along $Y=Y_1$ .The plate is L_X and L_Y in dimensions and $\xi = UT + {}^{\varepsilon}/_{2}$ as shown in figure1, also $B = B_X B_Y$, where $B_x =$

$$
\begin{aligned}
&1 - H\left(x - \xi + \frac{\varepsilon}{2}\right) \ldots\ldots\ldots 0 < 1 < \frac{\varepsilon}{2}\\
&H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \ldots\ldots\ldots \frac{\varepsilon}{2} < 1 < \frac{L_x}{u}\\
&H\left(\xi + \frac{\varepsilon}{2}\right) \ldots\ldots \frac{L_x}{u} \leq T < (L_x + \varepsilon)/U\\
&0 \ldots\ldots (L_x + \varepsilon)/U \leq T
\end{aligned}
$$

$$B_y = H\left(y - y_1 + \frac{\mu}{2}\right) - H\left(y - y_1 - \frac{\mu}{2}\right)$$

H (x) is the Heaviside function defined as

$$H(x) = \begin{cases} 1 & x>0 \\ 0.5 & x=0 \\ 0 & x<0 \end{cases}$$

K is the foundation stiffness, G is the foundation Shear modulus and M_f is the mass of the foundation.
The equations for the bending moments, twisting moments and Shear force are given as follows [2]:

$$M_x = -D\left(\frac{\partial \psi_x}{\partial x} + \boldsymbol{\upsilon}\frac{\partial \psi_y}{\partial y}\right) \tag{4}$$

$$M_y = -D\left(\frac{\partial \psi_y}{\partial y} + \boldsymbol{\upsilon}\frac{\partial \psi_x}{\partial x}\right) \tag{5}$$

$$M_{xy} = \left(\frac{1-\upsilon}{2}\right) D\left(\frac{\partial \psi_x}{\partial x} + \boldsymbol{\upsilon}\frac{\partial \psi_y}{\partial y}\right) \tag{6}$$

$$Qx = -K^2 G_1 h\left(\boldsymbol{\psi_x} - \frac{\partial W}{\partial x}\right) \tag{7}$$

$$Qy = -K2G_1 h\left(\boldsymbol{\psi_y} - \frac{\partial W}{\partial y}\right) \tag{8}$$

$$\frac{\partial W}{\partial T} = D_T \tag{9}$$

$$\frac{\partial W}{\partial x} = D_x \tag{10}$$

$$\frac{\partial W}{\partial y} = D_y \tag{11}$$

Where G_1 is the modulus of rigidity of the plate, D is the flexural rigidity of the plate defined by $D = \frac{1}{12}Eh^3 (1-\boldsymbol{\upsilon}^2) = G_1 h^3/6(1-\boldsymbol{\upsilon})$ for isotopic plate k^2 is the Shear correction factor and $\boldsymbol{\upsilon}$ is the Poisson's ratio of the plate.

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The set of partial differential equations (1) – (11), are the partial differential equations to be solved for the following eleven dependent variables: Q_x, Q_y, M_x, M_y, M_{xy}, $\psi_{x,t}$, $\psi_{y,t}$, $D_{t,}$ D_x, D_y and W. A numerical procedure, finite difference method, can be used to solve the system of equations, (1) - (11). after simplification [1,2]
Rearranging the resulting algebraic equations in matrix form gives:[2]

$$H_{i,j+1} S'_{i,j+1} + I_{i+1,j+i} S'_{i+1,j+1} = - G_{i,j}S'_{I,,j} - J_{i+1,j} S'_{i+1,j} + L_k \quad (12)$$

$$i = 1, 2, 3, ..., N-1; \ j = 1, 2, 3, ... M -1$$

Where N and M are the number of the nodal points along X and Y axes respectively.

$$L_k = K_{i, j} S^{\circ}_{i,j} + L_{i,j +i} , S^{\circ}_{i,j+1} M_{i+1} S^{\circ}_{i+1,j} + N_{i+1, j+1} +S^{\circ}_{i+1,j+1} + P_1 \quad (13)$$

Each term in equations (12) and (13) is an 11×11 matrix.

IV. The Shear, Rotatory and Kirchhoff Railway Bridges (Plates) Resting On Pasternak Foundation

In order to compare the effects of shear deformation and rotatory inertia on the deflection of the railway bridges under a moving railway vehicle (load) supported by a sub-grade (Pasternak foundation), the following types of plates are considered: the share plate (no rotatory inertia effect.), the rotatory plate (no shear deformation effect) and Kirchhoff plate (non – Mindlin plate).

V. Result Discussion

The numerical calculations were carried out for a simply supported rectangular plate (railway bridge) resting on a Pasternak foundation and subjected to a moving railway vehicle (load.). Damping effect was neglected. For a specific value of the parameters, deflection of the railway bridge calculated and plotted as a function of time. The following results were obtained: The Deflection of the railway bridge increases as K increases for various time t. (as we can see in figures 2, 3 and 4). The response maximum amplitude decreases with an increase in the value of G for fixed value of K, Arp and Up. (as we can see in figure 2 also). The response amplitude of the railway bridge continuously supported by a subgrade is less than that of the plate not resting on any elastic subgrade (i.e K=0, G=0). Also as K and G increases, the response amplitude decreases. (as we can see in figure 5). As Arp increases, the response maximum amplitude increases for fixed values of K and G. (as we can see in figure 6). For t < 0.24, the maximum amplitudes of the shear railway bridge decreases as velocity increases. Also for the same time range and fixed values of K, G and Arp, it is observed that the shear railway bridge has the largest value of the maximum amplitude for all the values of velocity considered. (as we can see in figure 7). Shear railway bridge has the maximum amplitude followed by Mindlin railway bridge then non-Mindlin railway bridge. Rotatory railway bridge has the least. It is also observed that as G increases, the maximum amplitude of the shear railway bridge decreases for fixed values of K and Arp.(as we can see in figures 8 and 10). Shear railway bridge produces the maximum deflection for fixed values of K, G, U and Arp. It is also observed that there is no clear cut difference between the deflection of non-Mindlin and rotatory railway bridge. In other words, the effect of rotatory inertia is minimal when compared with the effect shear deformation. (as we can see in figure 10). As Arp increases, the maximum amplitude response for both Mindlin and shear railway bridges increase. The increase in the maximum amplitude response for the cases of rotatory and non-Mindlin railway bridges are not obvious (as we can see in figure IX)

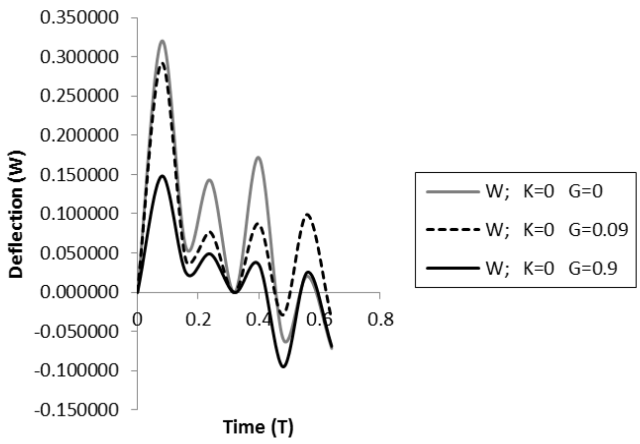

Fig. II. Deflection of the railway bridge for K=0, Arp=0.5 and various values of G and time t.

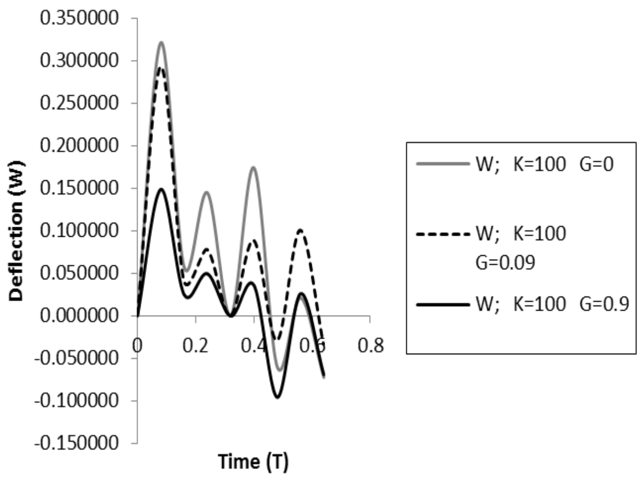

Fig. III. Deflection of the railway bridge for K=100, Arp=0.5 and various values of G and time t.

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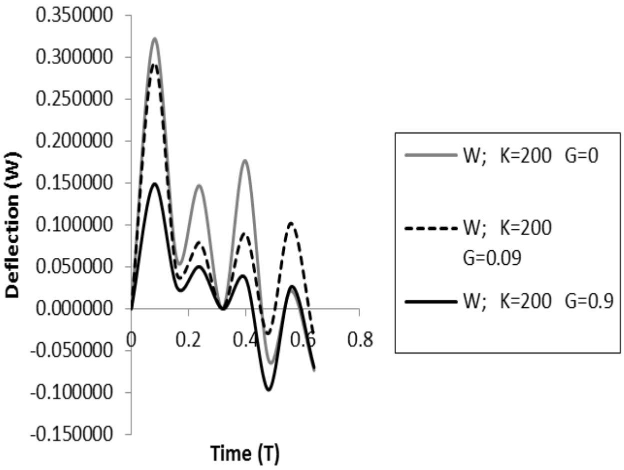

Fig. IV.. Deflection of the railway bridge for K=200, Arp=0.5, and various values of G and time t.

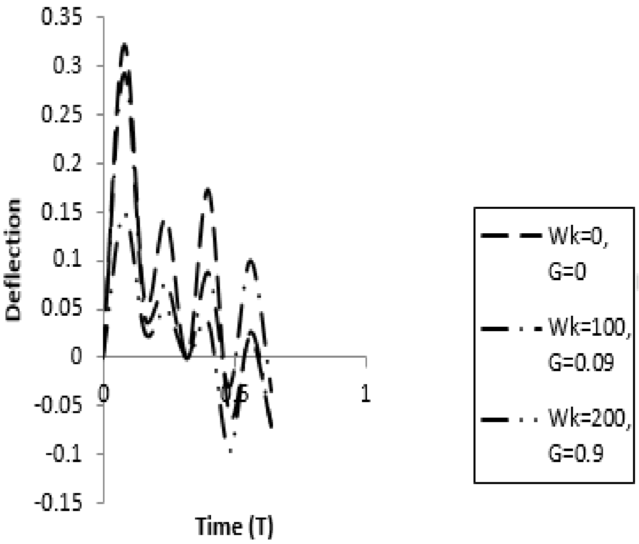

Fig. V. Deflection of the railway bridge at Arp=0.5 and different values of k, G for various values time

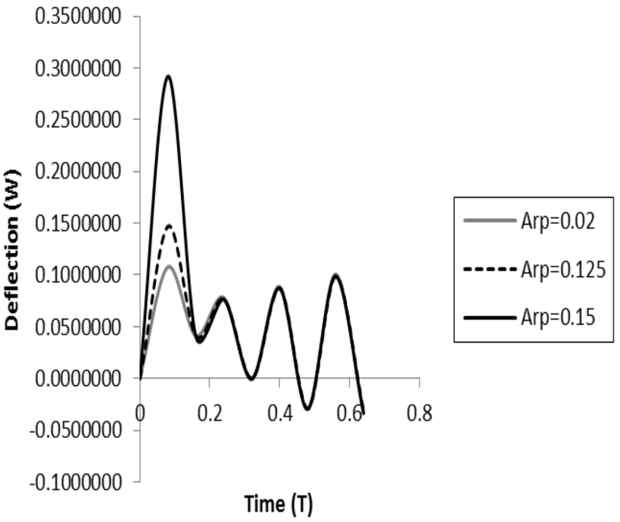

Fig. VI.. Deflection of the railway bridge for K=0, G=0.09 and various values of Arp and time

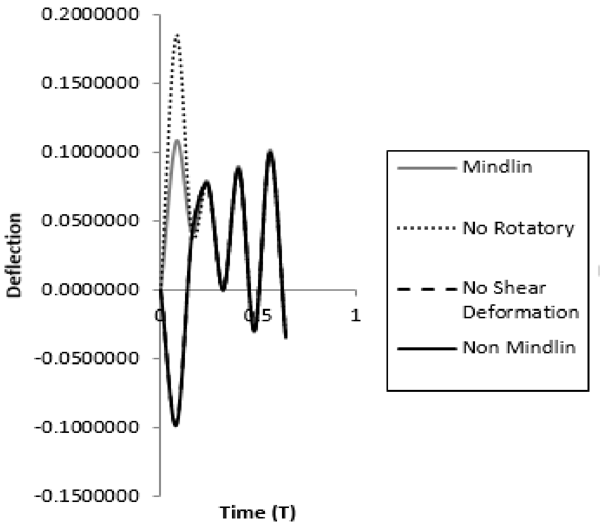

Fig. VII.. Deflection of the Mindlin, Non Mindlin, Rotatory, and Shear Railway bridges for K=100, G=0.09, Arp=0.02, U=1.5 and various values of time

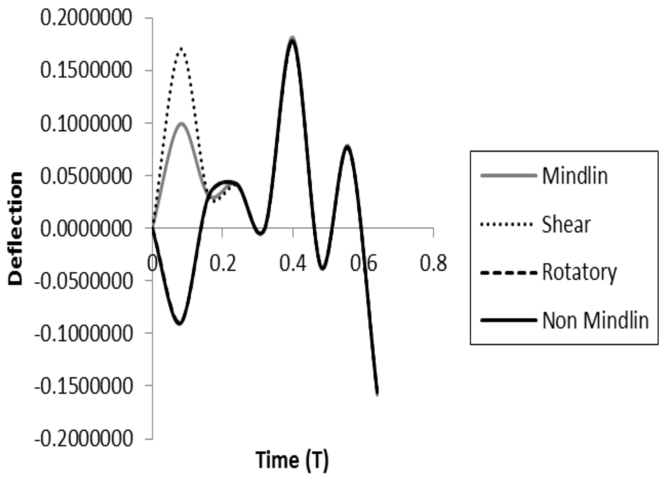

Fig VIII. Deflection of the Mindlin, Non Mindlin, Rotatory and Shear railway bridges for K=100, G=0.09, Arp=0.02, u=2.5 and various values of time

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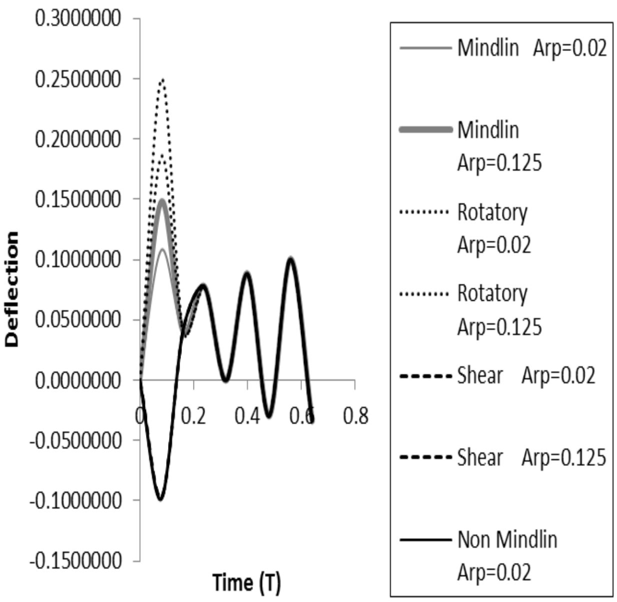

Fig, IX. Deflection of the Mindlin, Non Mindlin, Rotatory, and Shear Railway bridges at K=100, G=0.09, Up=1.5 and different values of 'Arp' and time

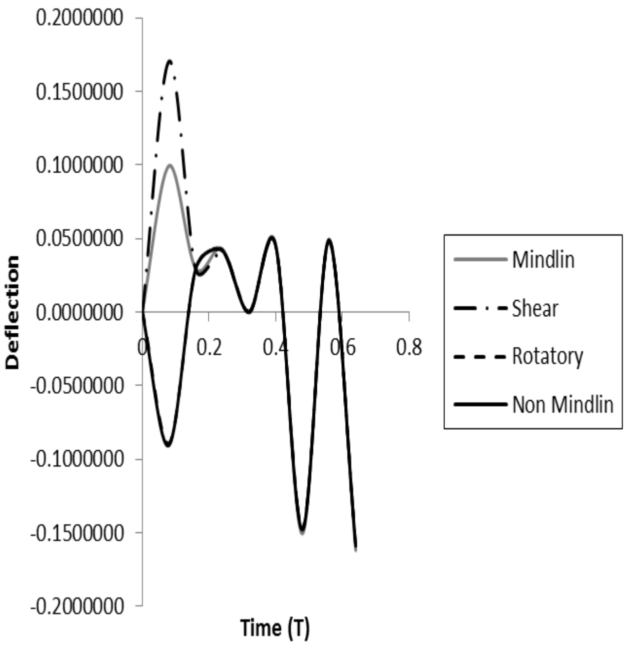

Fig. X. Deflection of the Mindlin, Non Mindlin, Rotatory and Shear Railway bridges for K=100, G=0.18, Arp=0.02, u=1.5 and various values of time

VI. Conclusion

The structure of interest was a railway bridge modelled as a Mindlin rectangular elastic railway bridge, on Pasternak foundation, under the influence of a uniform partially distributed moving railway vehicle. The problem was to determine the dynamic response of the whole system. Finite Difference technique was adopted in solving the resulting first order coupled partial differential equations obtained from the governing equations, for the simply supported railway bridge. The study has contributed to scientific knowledge by showing that elastic subgrade (Pasternak), on which the railway bridge rests has a significant effect on the dynamic response of the bridge to moving railway vehicle, modeled as a partially distributed moving load. The effect of rotatory inertia and shear deformation on the dynamic response of the railway bridge to the moving railway vehicle gives more realistic results for practical application especially when such railway bridge is considered to rest on a Pasternak foundation.

References

[1] J. A. Gbadeyan and M. C. Agarana, ''Dynamic Analysis of Railway bridges supported by Winkler Foundation under Uniform Partially Distributed Moving Railway vehicle'' COMPRAIL 2014, Rome, Italy Conference proceedings. Organised by Wessex Institute, Southampton, UK.

[2] J. A. Gbadeyan and M.S. Dada '' Dynamic Response of a Mindlin Elastic Rectangular Plate under a Distributed Moving Mass'' International Journal of Mechanical Science. Vol. 48: no Pp. 323 – 340. 2006

[3] E Molinear, P. Museros and M.D. Martinez-Rodrigo ''Retrofit of existing railway bridges of shut to medium spans for high-speed traffic using viscoelastic dampers'' Vol 40, no Pp 519-528. 2012

[4] A Sofi,'' Non-linear in-plane vibrations of inclined cables carrying moving oscillators.'' Journal of sound and vibration'', Vol. 332, no 7, Pp 1712-1724, 2013

[5] A. Nikkhoo, and F.R..Rofooei, '' Parametric Study of the dynamic response of the dynamic response of their rectangular plates traversed by a moving mass'', Acta Mechanica, Vol. 223, no.1, Pp.15-27, 2012

[6] , M.R Shadnam, M. Mofid and J.E. Akin, ''On the dynamic response of rectangular plate, with moving mass.'' Vol.39, no 9, Pp 797-806, 2001

[7] V. K Gerg and R.V Dukkipati ''Dynamics of Railway Vehicle Systems.'' Academic Press New York. 1984.

[8] R. D Mindlin ''Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates'', Journal of Applied Mechanics, Vol. 18, no , Pp 1957.

[9] O. Civalek, ''Large Deflection Static and Dynamic Analysis of Thin Circular Plates Resting On Two Parameter Elastic Foundation HDQ/FO Couple Methodology Approaches.'', International Journal of Computational Mechanics, Vol. 2, no 2, Pp. 271-291, 2005

[10] J. A. Gbadeyan and M.S. Dada.'' The Dynamic Response of Plates on Pasternak Foundation to Distributed Moving Load'', Journal of the Nigerian Association of Mathematical Physics. Vol.5, no Pp. 185 – 200, 2001

[11] N. D. Beskou and D. D. Theodorakopoulos, "Dynamic effects of moving loads on road pavements: a review," Soil Dynamics and Earthquake Engineering, vol. 31, no. 4, pp. 547–567, 2011. View at Publisher ·

[12] J. V. Amiri, A. Nikkhoo, M. R. Davoodi, and M. E. Hassanabadi, "Vibration analysis of a Mindlin elastic plate under a moving mass excitation by eigenfunction expansion method," Thin-Walled Structures, vol. 62, pp. 53–64, 2013.

[13] M. R. Shadnam, M. Mofid, and J. E. Akin, "On the dynamic response of rectangular plate, with moving mass," Thin-Walled Structures, vol. 39, no. 9, pp. 797–806, 2001.

[14] S. Wu, M.-L. Lee, and T.-S. Lai, "The Dynamic analysis of a flat plate under a moving load by the finite element method," International Journal for Numerical Methods in Engineering, vol. 24, no. 4, pp. 743–762, 1987.

[15] R.-T. Wang and T.-Y. Lin, "Random vibration of multi-span Mindlin plate due to moving load," Journal of the Chinese Institute of Engineers, vol. 21, no. 3, pp. 347–356, 1998.

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