# On modified Noor iterations for nonlinear equations in Banach spaces*

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**Abstract:** We introduce a new class of nonlinear mappings, the class of asymptotically $\phi$-hemicontractive mappings in the intermediate sense and approximate the unique common fixed point of a family of three of these mappings in Banach spaces. Our results improve and generalize the results of Xue and Fan [Zhiqun Xue, Ruiqin Fan, Some comments on Noor's iterations in Banach spaces, Applied Mathematics and Computation 206 (2008) 12-15] which in turn is a correction of the results of Rafiq [Arif Rafiq, Modified Noor iterations for nonlinear equations in Banach spaces, Applied Mathematics and Computation 182 (2006) 589-595].

**Keywords:** Noor iterative scheme with errors, Banach spaces, asymptotically $\phi$-hemicontractive mappings in the intermediate sense, common fixed point, $\phi$-strongly quasi-accretive in the intermediate sense.

## 1 Introduction

Let $E$ be a real Banach space, $D$ a nonempty subset of $E$ and $\phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We associate a $\phi$-normalized duality mapping $J_\phi : E \to 2^{E^*}$ to the function $\phi$

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defined by

$$J_\phi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\phi(\|x\|) \ \ and \ \ \|f^*\| = \phi(\|x\|)\}, \tag{1.1}$$

where $E^*$ denotes the dual space of $E$ and $\langle ., . \rangle$ denotes the duality pairing. We shall denote a single-valued duality mapping by $j_\phi$. If $\phi(t) = t$, then $J_\phi$ reduces to the usual duality mapping $J$.

The following relationship exists between $J_\phi$ and $J$, which can easily be shown.

$$J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|} J(x) \ \ \forall \ x \neq 0. \tag{1.2}$$

The following definition was given in [10].

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ and $F(T)$ be the nonempty set of fixed points of $T$.

**Definition 1.1.** [10]. $T$ is said to be asymptotically $\phi$-hemicontractive, if there exists a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x - y) \in J_\phi(x - y)$ such that for some $n_0 \in \mathbb{N}$

$$\langle T^n x - y, j_\phi(x - y) \rangle \leq k_n(\phi(\|x - y\|))^2 \ \ \forall \ \ x \in D(T), \ \ y \in F(T), \ n \geq n_0. \tag{1.3}$$

**Definition 1.2.** [21]. $T$ is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \leq 0. \tag{1.4}$$

Put

$$\tau_n = \max \left\{ 0, \sup_{x,y \in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \right\}. \tag{1.5}$$

It follows that $\tau_n \to 0$ as $n \to \infty$. Hence, (1.4) is reduced to the following:

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 + \tau_n, \forall n \geq 1, x, y \in C. \tag{1.6}$$

Qin *et al.* [21] recently introduced the class of asymptotically pseudocontractive mappings in the intermediate sense. We remark that if $\tau_n = 0 \ \ \forall n \geq 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically pseudocontractive mappings. Olaleru and Okeke [18] proved some strong convergence results of Noor type iteration for a uniformly $L$-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness.

Bruck *et al.* [3] in 1993 introduced the class of *asymptotically nonexpansive mappings in the intermediate sense* as follows.
The mapping $T : D \to D$ is said to be *asymptotically nonexpansive in the intermediate sense* provided $T$ is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in D} (\|T^n x - T^n y\| - \|x-y\|) \leq 0. \tag{1.7}$$

Motivated by the facts above, we introduce the following class of nonlinear mappings.

**Definition 1.3.** A mapping $T : D \to D$ is said to be *asymptotically $\phi$- hemicontractive mapping in the intermediate sense,* if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x-p) \in J_\phi(x-p)$ such that for some $n_0 \in \mathbb{N}$

$$\limsup_{n\to\infty} \sup_{x,p\in D\times F(T)} \big(\langle T^n x - p, j_\phi(x-p)\rangle - k_n(\phi(\|x-p\|))^2\big) \leq 0, \quad \forall\, n \geq n_0. \tag{1.8}$$

Put

$$\xi_n = \max\left\{0, \sup_{x,p\in D\times F(T)} \big(\langle T^n x - p, j_\phi(x-p)\rangle - k_n(\phi(\|x-p\|))^2\big)\right\}. \tag{1.9}$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Hence, (1.8) is reduced to the following

$$\langle T^n x - p, j_\phi(x-p)\rangle \leq (k_n+\xi_n)(\phi(\|x-p\|))^2, \quad \forall x \in D,\ p \in F(T),\ n \geq n_0. \tag{1.10}$$

If $\xi_n = 0$ for all $n \in \mathbb{N}$, then the class of asymptotically $\phi$- hemicontractive mappings in the intermediate sense reduces to the class of asymptotically $\phi$-hemicontractive mappings.

**Definition 1.4.** A mapping $A$ is called *$\phi$-strongly quasi-accretive in the intermediate sense* if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x-p) \in J_\phi(x-p)$ such that for some $n_0 \in \mathbb{N}$, $x \in D(A)$, $p \in N(A)$ and $\xi_n$ as defined in (1.10), then

$$\langle Ax - Ap, j_\phi(x-p)\rangle \geq (k_n+\xi_n)(\phi(\|x-p\|))^2. \tag{1.11}$$

The following definitions will be needed in this study.

**Definition 1.5.** [22]. A map $T : E \to E$ is called strongly accretive if there exists a constant $k > 0$ such that, for each $x, y \in E$, there is a $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1.12}$$

**Definition 1.6.** [22]. An operator $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called *strongly pseudocontractive* if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and a constant $0 < k < 1$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2. \tag{1.13}$$

The class of strongly accretive operators is closely related to the class of strongly pseudocontractive operators. It is well known that $T$ is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive, where $I$ denotes the identity operator. Browder [2] and Kato [9] indepedently introduced the concept of accretive operators in 1967. One of the early results in the theory of accretive operators credited to Browder states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0 \tag{1.14}$$

is solvable if $T$ is locally Lipschitzian and accretive in an appropriate Banach space. These class of operators have been studied extensively by several authors (see, e.g. [4], [5], [10], [20], [16], [24], [25]).

In 1953, Mann introduced the Mann iterative scheme and used it to prove the convergence of the sequence to the fixed points for which the Banach principle is not applicable. Later in 1974, Ishikawa [8] introduced an iterative process to obtain the convergence of a Lipschitzian pseudocontractive operator when Mann iterative scheme failed to converge. In 2000 Noor [11] gave the following three-step iterative scheme (or Noor iteration) for solving nonlinear operator equations in uniformly smooth Banach spaces.
Let $D$ be a nonempty convex subset of $E$ and let $T : D \to D$ be a mapping. For a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative schemes

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0 \end{cases} \tag{1.15}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying some conditions.

In 1989, Glowinski and Le-Tallec [6] used a three-step iterative process to solve elastoviscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge *et al.* [7] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [6] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions. Hence, we can conclude by observing that three-step iterative scheme play an important role in solving various problems in pure and applied sciences (see, e.g. [1], [12], [13], [15], [16], [17], [18], [23]).

Rafiq [22] recently introduced the following modified three-step iterative scheme and used it to approximate the unique common fixed point of a family of strongly pseudocontractive operators.

Let $T_1, T_2, T_3 : D \to D$ be three given mappings. For a given $x_0 \in D$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n \\ y_n = (1-\beta_n)x_n + \beta_n T_2 z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n, \quad n \geq 0, \end{cases} \tag{1.16}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences in $[0,1]$ satisfying some conditions.

Yang *et al.* [26] in 2010 introduced the following three step iterative scheme. Let $E$ be a normed space, $D$ be a nonempty convex subset of $E$. Let $T_i : D \to D$, $(i = 1, 2, 3)$ be given asymptotically nonexpansive mappings in the intermediate sense. Then for a given $x_1 \in D$ and $n \geq 1$, compute the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})x_n + a_{n1}T_1^n y_n + b_{n1}T_1^n z_n + e_{n1}T_1^n x_n + c_{n1}u_n, \\ y_n = (1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}T_2^n z_n + b_{n2}T_2^n x_n + c_{n2}v_n, \\ z_n = (1 - a_{n3} - c_{n3})x_n + a_{n3}T_3^n x_n + c_{n3}w_n, \end{cases} \tag{1.17}$$

where $\{a_{ni}\}$, $\{c_{ni}\}$, $\{b_{n1}\}$, $\{b_{n2}\}$, $\{e_{n1}\}$, $\{a_{n3}+c_{n3}\}$, $\{a_{n2}+b_{n2}+c_{n2}\}$ and $\{a_{n1}+b_{n1}+c_{n1}+e_{n1}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in $D$. The iterative schemes (1.17) are called the modified three-step iterations with errors.

If $T_1 = T_2 = T_3 = T$ and $b_{n1} = b_{n2} = c_{n1} = c_{n2} = c_{n3} = e_{n1} \equiv 0$, then (1.17) reduces to the Noor iteration defined in [11]. If $b_{n1} = e_{n1} = c_{n1} = b_{n2} = c_{n2} = c_{n3} \equiv 0$, then (1.17) reduces to (1.16). This means that the modified Noor iterative scheme introduced by Rafiq [18] is a special case of the modified three-step iterations with errors introduced by Yang *et al.* [26].

Rafiq [22] in 2006 introduced the modified Noor iterative scheme (1.16) and obtained some fixed point results for a family of three strongly pseudocontractive self maps in Banach spaces. However, His proof was incorrect. Xue and Fan [25] in 2008 obtained the corrected results for Rafiq [22].

In this study, we approximate the common fixed points of a family of three asymptotically $\phi$-hemicontractive mappings in the intermediate sense using the three step iterative scheme (1.17) introduced by Yang *et al.* [26]. Our results improves and generalizes the results of Kim and Lee [10], Xue and Fan [25], Yang *et al.* [26] and several others in literature.

The following lemmas will be needed in this study.

**Lemma 1.1.** [10]. Let $J_\phi : E \to 2^{E^*}$ be a $\phi$-normalized duality mapping. Then for any $x, y \in E$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\frac{\|x+y\|}{\phi(\|x+y\|)}\langle y, j_\phi(x+y)\rangle \quad \forall\ j_\phi(x+y) \in J_\phi(x+y).$$

We remark that if $\phi$ is an identity, then we have the following inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \forall \quad j(x+y) \in J(x+y).$$

**Lemma 1.2.** [24]. Let $\{\rho\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality:

$$\rho_{n+1} \leq (1-\lambda_n)\rho_n + \sigma_n, \quad n \geq 0,$$

where $\lambda_n \in (0,1)$, $n = 0, 1, 2, \cdots, \sum_{n=0}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

# 2 Main Results

**Theorem 2.1.** Let $E$ be a real Banach space and $D$ be a nonempty closed convex subset of $E$. Let $T_1, T_2$ and $T_3$ be asymptotically $\phi$-hemicontractive mappings in the intermediate sense self maps of $D$ with $T_1(D)$ bounded and $T_1, T_2$ and $T_3$ uniformly continuous. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.17), where $\{a_{ni}\}$, $\{c_{ni}\}$, $\{b_{n1}\}$, $\{b_{n2}\}$, $\{e_{n1}\}$, $\{a_{n3}+c_{n3}\}$, $\{a_{n2}+b_{n2}+c_{n2}\}$ and $\{a_{n1}+b_{n1}+c_{n1}+e_{n1}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in $D$ satisfying the conditions: $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} a_{n1} = \infty$. If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of $T_1, T_2$ and $T_3$.

**Proof.** Since $T_1, T_2, T_3$ are asymptotically $\phi$-hemicontractive mappings in the intermediate sense, there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x-p) \in J_\phi(x-p)$ such that for some $n_0 \in \mathbb{N}$

$$\langle T_i^n x - p, j_\phi(x-p)\rangle \leq (k_n+\xi_n)(\phi(\|x-p\|))^2, \ \forall x \in D,\ p \in F(T),\ n \geq n_0, i = 1, 2, 3. \tag{2.1}$$

where $\{\xi_n\}$ and $\{k_n\}$ is as defined in (1.10). Let $p \in F(T_1) \cap F(T_2) \cap F(T_3)$ and

$$\begin{aligned} M_1 &= \|x_0 - p\| + \sup_{n\geq 0} \|T_1^n y_n - p\| + \sup_{n\geq 0} \|T_1^n z_n - p\| \\ &\quad + \sup_{n\geq 0} \|T_1^n x_n - p\| + \sup_{n\geq 0} \|u_n - p\|. \end{aligned} \tag{2.2}$$

Clearly, $M_1$ is finite. We now show that $\{x_n - p\}_{n\geq 0}$ is also bounded. Observe that $\|x_0 - p\| \leq M_1$. It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})(x_n - p) + a_{n1}(T_1^n y_n - p) \\ &\quad + b_{n1}(T_1^n z_n - p) + e_{n1}(T_1^n x_n - p) + c_{n1}(u_n - p)\| \\ &\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})\|x_n - p\| + a_{n1}\|T_1^n y_n - p\| \\ &\quad + b_{n1}\|T_1^n z_n - p\| + e_{n1}\|T_1^n x_n - p\| + c_{n1}\|u_n - p\| \\ &\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})M_1 + a_{n1}M_1 + b_{n1}M_1 \\ &\quad + e_{n1}M_1 + c_{n1}M_1 \\ &= M_1, \end{aligned} \tag{2.3}$$

using the uniform continuity of $T_3$, we obtain that $\{T_3^n x_n\}$ is bounded. Denote

$$M_2 = \max\left\{M_1, \sup_{n\geq 0}\{\|T_3^n x_n - p\|\}, \sup_{n\geq 0}\{\|w_n - p\|\}\right\}, \tag{2.4}$$

then we have:

$$\begin{aligned} \|z_n - p\| &\leq (1 - a_{n3} - c_{n3})\|x_n - p\| + a_{n3}\|T_3^n x_n - p\| + c_{n3}\|w_n - p\| \\ &\leq (1 - a_{n3} - c_{n3})M_1 + a_{n3}M_2 + c_{n3}M_2 \\ &\leq (1 - a_{n3} - c_{n3})M_2 + a_{n3}M_2 + c_{n3}M_2 \\ &= M_2. \end{aligned} \tag{2.5}$$

Recall that $T_2$ is uniformly continuous, so that $\{T_2^n z_n\}$ is bounded. Let

$$M = \sup_{n\geq 0} \|T_2^n z_n - p\| + \sup_{n\geq 0} \|T_2^n x_n - p\| + \sup_{n\geq 0} \|v_n - p\| + M_2,$$

then $M$ is finite. Since $\{x_n - p\}_{n\geq 0}$ is bounded and $\phi$ is a continuous strictly increasing function, $M^* := \sup_{n\geq 0} \phi(\|x_{n+1} - p\|)$ is also finite. Using Lemma 1.1, (1.17) and (2.1), then for $n \geq 0$ and $j_\phi(x_{n+1} - p) \in J(x_{n+1} - p)$, we have:

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-a_{n1}-b_{n1}-c_{n1}-e_{n1})(x_n-p)+a_{n1}(T_1^n y_n-p)\\
&\quad +b_{n1}(T_1^n z_n-p)+e_{n1}(T_1^n x_n-p)+c_{n1}(u_n-p)\|^2\\
&\le (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2\\
&\quad +2\langle a_{n1}(T_1^n y_n-p)+b_{n1}(T_1^n z_n-p)+e_{n1}(T_1^n x_n-p)\\
&\quad +c_{n1}(u_n-p), \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\rangle\\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2\\
&\quad +2a_{n1}\left\langle T_1^n y_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle\\
&\quad +2b_{n1}\left\langle T_1^n z_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle\\
&\quad +2e_{n1}\left\langle T_1^n x_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle\\
&\quad +2c_{n1}\left\langle u_n-p, \tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)} j_\phi(x_{n+1}-p)\right\rangle\\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2\\
&\quad +2a_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n y_n-T_1^n x_{n+1}+T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2b_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n z_n-T_1^n x_{n+1}+T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2e_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_n-T_1^n x_{n+1}+T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2c_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle u_n-p, j_\phi(x_{n+1}-p)\rangle\\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2\\
&\quad +2a_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n y_n-T_1^n x_{n+1}, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2a_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2b_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n z_n-T_1^n x_{n+1}, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2b_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2e_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_n-T_1^n x_{n+1}, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2e_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle T_1^n x_{n+1}-p, j_\phi(x_{n+1}-p)\rangle\\
&\quad +2c_{n1}\tfrac{\|x_{n+1}-p\|}{\phi(\|x_{n+1}-p\|)}\langle u_n-p, j_\phi(x_{n+1}-p)\rangle\\
&\le (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}\|x_{n+1}-p\|\|T_1^n y_n-T_1^n x_{n+1}\|\\
&\quad +2a_{n1}(k_n+\xi_n)\|x_{n+1}-p\|\phi(\|x_{n+1}-p\|)+2b_{n1}\|x_{n+1}-p\|\|T_1^n z_n-T_1^n x_{n+1}\|\\
&\quad +2b_{n1}(k_n+\xi_n)\|x_{n+1}-p\|\phi(\|x_{n+1}-p\|)+2e_{n1}\|x_{n+1}-p\|\|T_1^n x_n-T_1^n x_{n+1}\|\\
&\quad +2e_{n1}(k_n+\xi_n)\|x_{n+1}-p\|\phi(\|x_{n+1}-p\|)+2c_{n1}\|x_{n+1}-p\|\|u_n-p\|\\
&\le (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}\|x_{n+1}-p\|\|T_1^n y_n-T_1^n x_{n+1}\|\\
&\quad +2a_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|+2b_{n1}\|x_{n+1}-p\|\|T_1^n z_n-T_1^n x_{n+1}\|\\
&\quad +2b_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|+2e_{n1}\|x_{n+1}-p\|\|T_1^n x_n-T_1^n x_{n+1}\|\\
&\quad +2e_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|+2c_{n1}\|x_{n+1}-p\|\|u_n-p\|\\
&\le (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|\\
&\quad +2b_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|+2e_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|\\
&\quad +2M_1\{a_{n1}\|T_1^n y_n-T_1^n x_{n+1}\|+b_{n1}\|T_1^n z_n-T_1^n x_{n+1}\|\\
&\quad +e_{n1}\|T_1^n x_n-T_1^n x_{n+1}\|+c_{n1}\|u_n-p\|\}\\
&= (1-a_{n1}-b_{n1}-c_{n1}-e_{n1})^2\|x_n-p\|^2+2a_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|\\
&\quad +2b_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|+2e_{n1}(k_n+\xi_n)M^*\|x_{n+1}-p\|+2\delta_n, \qquad (2.6)
\end{aligned}
$$

where

$$\begin{aligned}\delta_n &= M_1\{a_{n1}\|T_1^n y_n - T_1^n x_{n+1}\| + b_{n1}\|T_1^n z_n - T_1^n x_{n+1}\| \\ &\quad + e_{n1}\|T_1^n x_n - T_1^n x_{n+1}\| + c_{n1}\|u_n - p\|\}. \qquad (2.7)\end{aligned}$$

Using (1.17), we have

$$\begin{aligned}\|y_n - x_{n+1}\| &= \|y_n - x_n + x_n - x_{n+1}\| \\ &\leq \|y_n - x_n\| + \|x_n - x_{n+1}\| \\ &= \|(1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}T_2^n z_n + b_{n2}T_2^n x_n + c_{n2}v_n - x_n\| \\ &\quad + \|x_n - \{(1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})x_n + a_{n1}T_1^n y_n + b_{n1}T_1^n z_n \\ &\quad + e_{n1}T_1^n x_n + c_{n1}u_n\}\| \\ &= \| - a_{n2}(x_n - T_2^n z_n) - b_{n2}(x_n - T_2^n x_n) - c_{n2}(x_n - v_n)\| \\ &\quad + \|a_{n1}(x_n - T_1^n y_n) + b_{n1}(x_n - T_1^n z_n) + c_{n1}(x_n - u_n) \\ &\quad + e_{n1}(x_n - T_1^n x_n)\| \\ &= \| - a_{n2}(x_n - p + p - T_2^n z_n) - b_{n2}(x_n - p + p - T_2^n x_n) \\ &\quad - c_{n2}(x_n - p + p - v_n)\| + \|a_{n1}(x_n - p + p - T_1^n y_n) \\ &\quad + b_{n1}(x_n - p + p - T_1^n z_n) + c_{n1}(x_n - p + p - u_n) \\ &\quad + e_{n1}(x_n - p + p - T_1^n x_n)\| \\ &\leq a_{n2}\|x_n - p\| + a_{n2}\|p - T_2^n z_n\| + b_{n2}\|x_n - p\| + b_{n2}\|p - T_2^n x_n\| \\ &\quad + c_{n2}\|x_n - p\| + c_{n2}\|p - v_n\| + a_{n1}\|x_n - p\| + a_{n1}\|p - T_1^n y_n\| \\ &\quad + b_{n1}\|x_n - p\| + b_{n1}\|p - T_1^n z_n\| + c_{n1}\|x_n - p\| + c_{n1}\|p - u_n\| \\ &\quad + e_{n1}\|x_n - p\| + e_{n1}\|p - T_1^n x_n\| \\ &\leq 2Ma_{n2} + 2Mb_{n2} + 2Mc_{n2} + 2Ma_{n1} + 2Mb_{n1} + 2Mc_{n1} + 2Me_{n1} \\ &= 2M\left(a_{n2} + b_{n2} + c_{n2} + a_{n1} + b_{n1} + c_{n1} + e_{n1}\right). \qquad (2.8)\end{aligned}$$

Using the condition that $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$, we obtain from (2.8)

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \qquad (2.9)$$

Using the uniform continuity of $T_1$, we have

$$\lim_{n\to\infty} \|T_1^n y_n - T_1^n x_{n+1}\| = 0. \qquad (2.10)$$

Similarly, $\lim_{n\to\infty} \|T_1^n z_n - T_1^n x_{n+1}\| = \lim_{n\to\infty} \|T_1^n x_n - T_1^n x_{n+1}\| = 0$. Hence, we have that $\lim_{n\to\infty} \delta_n = 0$.

Furthermore, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})(x_n - p) + a_{n1}(T_1^n y_n - p) \\ &\quad + b_{n1}(T_1^n z_n - p) + e_{n1}(T_1^n x_n - p) + c_{n1}(u_n - p)\| \\ &\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})\|x_n - p\| + a_{n1}\|T_1^n y_n - p\| \\ &\quad + b_{n1}\|T_1^n z_n - p\| + e_{n1}\|T_1^n x_n - p\| + c_{n1}\|u_n - p\| \\ &\leq (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})\|x_n - p\| \\ &\quad + (a_{n1} + b_{n1} + e_{n1} + c_{n1})M. \qquad (2.11)\end{aligned}$$

Since $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$, for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $(a_{n1} + b_{n1} + c_{n1} + e_{n1}) \leq \epsilon$ for all $n \geq k$. Let $\{t_n\} = \{a_{n1} + b_{n1} + c_{n1} + e_{n1}\}$. Substituting (2.11) into (2.6), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-t_n)^2\|x_n - p\|^2 + 2M^*(k_n + \xi_n)(a_{n1} + b_{n1} + e_{n1})\|x_{n+1} - p\| \\
&\quad +2\delta_n \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2M^*(k_n + \xi_n)(a_{n1} + b_{n1} + e_{n1}) \\
&\quad \times\{(1-t_n)^2\|x_n - p\| + t_n M\} + 2\delta_n \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2M^* t_n(k_n + \xi_n)\{(1-t_n)\|x_n - p\| + t_n M\} \\
&\quad +2\delta_n \\
&= (1-t_n)^2\|x_n - p\|^2 + 2M^* t_n(k_n + \xi_n)(1-t_n)\|x_n - p\| \\
&\quad +2MM^* t_n^2(k_n + \xi_n) + 2\delta_n \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2M^* t_n(k_n + \xi_n)(1-t_n) \\
&\quad \times\{(1-t_{n-1})\|x_{n-1} - p\| + t_{n-1}M\} + 2[MM^* t_n^2(k_n + \xi_n) + \delta_n] \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2M^*(k_n + \xi_n)t_n(1-t_n)(1-t_{n-1})\|x_{n-1} - p\| \\
&\quad +2[M^*(k_n + \xi_n)t_n(1-t_n)t_{n-1}M + MM^* t_n^2(k_n + \xi_n) + \delta_n] \\
&= (1-t_n)^2\|x_n - p\|^2 + 2M^* t_n(k_n + \xi_n)(1-t_n)(1-t_{n-1})\|x_{n-1} - p\| \\
&\quad +2\left[MM^*(k_n + \xi_n)t_n\{(1-t_n)t_{n-1} + t_n\} + \delta_n\right] \\
&\leq \cdots \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2t_n(k_n + \xi_n)\prod_{j=k}^{n}(1-t_j)M^*\|x_k - p\| \\
&\quad +2\{t_n^2 MM^*(k_n + \xi_n) \\
&\quad +t_n(k_n + \xi_n)MM^*\sum_{j=k}^{n-1}\left(t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right) + \delta_n\} \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2\{t_n^2(k_n + \xi_n)\prod_{j=k}^{n}(1-t_j)MM^* \\
&\quad +t_n^2 MM^*(k_n + \xi_n) \\
&\quad +t_n(k_n + \xi_n)MM^*\sum_{j=k}^{n-1}\left(t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right) + \delta_n\} \\
&\leq (1-t_n)^2\|x_n - p\|^2 + 2\theta_n, \qquad (2.12)
\end{aligned}$$

where

$$\begin{aligned}
\theta_n &= \left[t_n\prod_{j=k}^{n}(1-t_j) + t_n + \sum_{j=k}^{n-1}\left(t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right)\right] t_n(k_n + \xi_n)MM^* \\
&\quad +\delta_n. \qquad (2.13)
\end{aligned}$$

Observe that $\{\theta_n\}_{n\geq 0}$ converges to 0 as $n \to \infty$. Clearly, $\prod_{j=k}^{n}(1-t_j) \leq e^{-\sum_{j=k}^{n} t_j} \longrightarrow 0$ as $n \to \infty$ and

$$\sum_{j=k}^{n-1}\left\{t_{n-1-j}\prod_{j=k}^{n-1}(1-t_{n-j})\right\} \leq \sum_{j=k}^{n-1}\epsilon \to 0$$

as $\epsilon \to 0$. Let $\rho_n = \|x_n - p\|^2$, $\lambda_n = t_n$ and $\sigma_n = 2\theta_n$. Using the fact that $\lim_{n\to\infty}\theta_n = \lim_{n\to\infty}\delta_n = 0$ and Lemma 1.2, we have from (2.12) that

$$\lim_{n\to\infty}\|x_n - p\| = 0. \qquad (2.14)$$

The proof of Theorem 2.1 is completed. □

**Remark 2.2.** Theorem 2.1 improves and generalizes the results of Yang *et al.* [26], Xue and Fan [25] which in turn is a correction of the results of Rafiq [22].

**Theorem 2.3.** Let $E$ be a real Banach space, $T_1, T_2, T_3 : E \to E$ be uniformly continuous and *$\phi$-strongly quasi-accretive in the intermediate sense* operators with $R(I - T_1)$ bounded, where $I$ is the identity mapping on $E$. Let $p$ denote the unique common solution to the equation $T_i x = f,\ (i = 1, 2, 3)$. For a given $f \in E$, define the operator $H_i : E \to E$ by $H_i x = f + x - T_i x,\ (i = 1, 2, 3)$. For any $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} x_{n+1} = (1 - a_{n1} - b_{n1} - c_{n1} - e_{n1})x_n + a_{n1}H_1 y_n + b_{n1}H_1 z_n + e_{n1}H_1 x_n + c_{n1}u_n, \\ y_n = (1 - a_{n2} - b_{n2} - c_{n2})x_n + a_{n2}H_2 z_n + b_{n2}H_2 x_n + c_{n2}v_n, \\ z_n = (1 - a_{n3} - c_{n3})x_n + a_{n3}H_3 x_n + c_{n3}w_n, \end{cases} \tag{2.15}$$

where $\{a_{ni}\}$, $\{c_{ni}\}$, $\{b_{n1}\}$, $\{b_{n2}\}$, $\{e_{n1}\}$, $\{a_{n3}+c_{n3}\}$, $\{a_{n2}+b_{n2}+c_{n2}\}$ and $\{a_{n1}+b_{n1}+c_{n1}+e_{n1}\}$ are appropriate sequences in [0,1] for $i = 1, 2, 3$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in $D$ satisfying the conditions: $\{a_{n1}\}, \{a_{n2}\}, \{b_{n1}\}, \{b_{n2}\}, \{c_{n1}\}, \{c_{n2}\}, \{e_{n1}\} \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} a_{n1} = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common solution to $T_i x = f\ (i = 1, 2, 3)$.

**Proof.** Clearly, if $p$ is the unique common solution to the equation $T_i x = f\ (i = 1, 2, 3)$, it follows that $p$ is the unique common fixed point of $H_1, H_2$ and $H_3$. Using the fact that $T_1, T_2$ and $T_3$ are all $\phi$-strongly quasi-accretive in the intermediate sense operators, then $H_1, H_2$ and $H_3$ are all asymptotically $\phi$-hemicontractive mappings in the intermediate sense. Since $T_i\ (i = 1, 2, 3)$ is uniformly continuous with $R(I - T_1)$ bounded, this implies that $H_i\ (i = 1, 2, 3)$ is uniformly continuous with $R(H_1)$ bounded. Hence, Theorem 2.3 follows from Theorem 2.1. □

**Remark 2.4.** Theorem 2.3 improves and extends Theorem 2.2 of Xue and Fan [25] which in turn is a correction of the results of Rafiq [22].

**Example 2.5.** Let $E = (-\infty, +\infty)$ with the usual norm and let $D = [0, +\infty)$. We define $T_1 : D \to D$ by $T_1 x := \frac{x}{2(1+x)}$ for each $x \in D$. Hence, $F(T_1) = \{0\}$, $R(T_1) = [0, \frac{1}{2})$ and $T_1$ is a uniformly continuous and asymptotically $\phi$-hemicontractive mapping in the intermediate sense. Define $T_2 : D \to D$ by $T_2 x := \frac{x}{4}$ for all $x \in D$. Hence, $F(T_2) = \{0\}$ and $T_2$ is a uniformly continuous and strongly pseudocontractive mapping. Define $T_3 : D \to D$ by $T_3 x := \frac{\sin^4 x}{4}$ for each $x \in D$. Then $F(T_3) = \{0\}$ and $T_3$ is a uniformly continuous and asymptotically $\phi$-hemicontractive mapping in the intermediate sense. Set $\{a_{ni}\} = \{c_{ni}\} = \frac{1}{n^4}$,

$\{b_{n1}\} = \{e_{n1}\} = \{b_{n2}\} = \frac{1}{n^3}$, $\{k_n\} = 1$, $\{\xi_n\} = \frac{1}{n^2}$ for all $n \geq 0$ and $\phi(t) = \frac{t^2}{2}$ for each $t \in (-\infty, +\infty)$. Clearly, $F(T_1) \cap F(T_2) \cap F(T_3) = \{0\} = p \neq \emptyset$. For an arbitrary $x_0 \in D$, the sequence $\{x_n\}_{n=0}^{\infty} \subset D$ defined by (1.17) converges strongly to the common fixed point of $T_1, T_2$ and $T_3$ which is $\{0\}$, satisfying Theorem 2.1. This means that Theorem 2.1 is applicable.

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# References

[1] H. Akewe, G. A. Okeke, A. F. Olayiwola, *Strong convergence and stability of Kirk-multistep-type iterative schemes for contractive-type operators,* Fixed Point Theory and Applications 2014, 2014:45, 24 pages.

[2] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive in Banach space,* Bull. Amer. Math. Soc. 73 (1967), 875-882.

[3] R. Bruck, T. Kuczumow, S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property,* Colloquium Mathematicum, Vol. LXV (1993), 169-179.

[4] S. S. Chang, Y. J. Cho, B. S. Lee, S. H. Kang, *Iterative approximation of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces,* J. Math. Anal. Appl. 224 (1998), 165-194.

[5] L. J. Ciric, J. S. Ume, *Ishikawa iteration process for strongly pseudocontractive operator in arbitrary Banach space,* Commun. 8 (2003), 43-48.

[6] R. Glowinski, P. Le-Tallec, *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics,* SIAM, Philadelphia, 1989.

[7] S. Haubruge, V. H. Nguyen, J. J. Strodiot, *Convergence analysis and applications of the Glowinski-Le-Tallec splitting method for finding a zero of the sum of two maximal monotone operators,* J. Optim. Theory Appl. 97 (1998), 645-673.

[8] S. Ishikawa, *Fixed points by a new iteration method,* Proc. Amer. Math. Soc. 44(1974), 147-150.

[9] T. Kato, *Nonlinear semigroup and evolution equations,* J. Math. Soc. Jpn. 19 (1967), 508-520.

[10] S. H. Kim, B. S. Lee, *A new approximation scheme for fixed points of asymptotically $\phi$-hemicontractive mappings,* Commun. Korean Math. Soc. 27 (2012), No. 1, pp. 167-174.

[11] M. A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl. 251(2000), 217-229.

[12] M. A. Noor, *Three-step iterative algorithms for multi-valued quasi variational inclusions,* J. Math. Anal. Appl. 255(2001), 589-604.

[13] M. A. Noor, *Some developments in general variational inequalities,* Appl. Math. Computation, 152 (2004), 199-277.

[14] M. A. Noor, T. M. Rassias, Z. Y. Huang, *Three-step iterations for nonlinear accretive operator equations,* J. Math. Anal. Appl. 274(2002), 59-68.

[15] G. A. Okeke, J. O. Olaleru, *Common fixed points of a three-step iteration with errors of asymptotically quasi-nonexpansive nonself-mappings in the intermediate sense in Banach spaces,* Fasciculi Mathematici, Nr 52, 2014, 93-115.

[16] G. A. Okeke, J. O. Olaleru, *Modified Noor iterations with errors for nonlinear equations in Banach spaces,* J. Nonlinear Sci. Appl. 7 (2014), 180-187.

[17] G. A. Okeke, J. O. Olaleru, *Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense for the modified Noor iterative scheme,* Mathematical Modelling & Computations, Vol. 05, No. 01, 2015, 15-28.

[18] J. O. Olaleru, G. A. Okeke, *Strong convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense,* British Journal of Mathematics & Computer Science, 2(3): (2012), 151-162.

[19] M. O. Osilike, Y. Shehu, *Cyclic algorithm for common fixed points of finite family of strictly pseudocontractive mappings of Browder-Petryshyn type,* Nonlinear Analysis 70 (2009) 3575-3583.

[20] M. O. Osilike, S. C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings,* Mathematical and Computer Modelling 32 (2000) 1181-1191.

[21] X. Qin, S. Y. Cho, J. K. Kim, *Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense,* Fixed Point Theory and Applications, Volume 2010, Article ID 186874, 14 pages.

[22] A. Rafiq, *Modified Noor iterations for nonlinear equations in Banach spaces,* Applied Mathematics and Computation 182 (2006), 589-595.

[23] G. S. Saluja, M. Postolache, A. Kurdi, *Convergence of three-step iterations for nearly asymptotically nonexpansive mappings in $CAT(k)$ space,* Journal of Inequalities and Applications (2015) 2015:156, 18 pages.

[24] X. Weng, *Fixed point iteration for local strictly pseudo-contractive mapping,* Proc. Amer. Math. Soc. 113(3) (1991), 727-731.

[25] Z. Xue, R. Fan, *Some comments on Noor's iterations in Banach spaces,* Applied Mathematics and Computation 206 (2008), 12-15.

[26] L-p. Yang, X. Xie, S. Peng, G. Hu, *Demiclosed principle and convergence for modified three step iterative process with errors of non-Lipschitzian mappings,* Journal of Computational and Applied Mathematics 234 (2010) 972-984.