The Burr X-Exponential Distribution: Theory and Applications

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***Abstract*— In this research, the Burr X-Exponential distribution was defined and explored using the Burr X family of distributions. Its basic statistical properties were identified and the method of maximum likelihood was proposed in estimating the model parameters. The model was applied to three different real data sets to assess its flexibility over its baseline distribution.**

***Index Terms*— Burr X distribution, Burr X family, Exponential distribution, Properties**

I INTRODUCTION

The Burr distribution has different forms, of these; the Burr-Type X and XII distributions have both received appreciable usage in probability distribution theory. Interestingly, the Burr-Type X distribution is related to some well-known standard theoretical distributions like the Weibull distribution and Gamma distribution.

The cdf and pdf of the Burr X distribution are given by;

$$F(x)=\left[1-e^{-x^2}\right]^{\theta} \quad (1)$$

and

$$f(x)=2\theta x e^{-x^2}\left[1-e^{-x^2}\right]^{\theta-1} \quad (2)$$

respectively

for $x>0, \theta\geq 0$

where; θ is the scale parameter.

Recently, the Burr X distribution has been used as a generator of other compound distributions by Yousof et al., (2016). This new family of distribution has been used to extend the Weibull and Lomax distributions. An application to real life data indicates that the Bur X-Lomax distribution is more flexible than the Lomax and other competing distributions.

There are other generalized families of distributions like the Beta-G (Eugene at al., 2002), Kumaraswamy-G (Cordeiro and de Castro, 2011), Weibull-G (Bourguinon et al., 2014), Weibull-X (Alzaatreh et al., 2013), Transmuted-G (Shaw and Buckley, 2007), Logistic-X (Tahir et al., 2016), Marshall-Olkin-G family of distributions (Marshall and Olkin, 1997) and many others, but of interest to us in this research is the Burr X-family of distributions.

The cdf and pdf of the Burr X family of distribution is given by;

$$F(x)=\left\{1-\exp\left[-\left(\frac{G(x)}{1-G(x)}\right)^2\right]\right\}^{\theta} \quad (3)$$

and

$$f(x)=\frac{2\theta g(x)G(x)}{\left(1-G(x)\right)^3}\exp\left[-\left(\frac{G(x)}{1-G(x)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{G(x)}{1-G(x)}\right)^2\right]\right\}^{\theta-1} \quad (4)$$

respectively.

for $x>0, \theta>0$

where; θ is a shape parameter whose role is to vary tail weight.

$G(x)$ and $g(x)$ are the cdf and pdf of the baseline distribution respectively.

This research is aimed at studying and exploring the Burr X-Exponential distribution using the family of distribution defined in (3) and (4) respectively. In the next section, the densities and properties of the Burr X-Exponential distribution are derived.

II The Burr X-Exponential Distribution

Consider a random variable X with a cdf and pdf defined by;

$$G(x)=1-\exp(-\lambda x) \quad (5)$$

and

$$g(x)=\lambda\exp(-\lambda x) \quad (6)$$

respectively.

for $x>0, \lambda>0$

where; λ is a scale parameter

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Then, the cdf of the Burr X-Exponential distribution is derived by substituting equation (5) into Equation (3) to give;

$$F(x)=\left\{1-\exp\left[-\left(\frac{1-\exp(-\lambda x)}{\exp(-\lambda x)}\right)^2\right]\right\}^\theta \tag{7}$$

For $x>0,\theta>0,\lambda>0$

Its corresponding pdf is given by;

$$f(x)=\frac{2\theta\lambda\left[1-e^{(-\lambda x)}\right]}{\left(e^{(-\lambda x)}\right)^2}\exp\left[-\left(\frac{1-e^{(-\lambda x)}}{e^{(-\lambda x)}}\right)^2\right]\times$$
$$\left\{1-\exp\left[-\left(\frac{1-e^{(-\lambda x)}}{e^{(-\lambda x)}}\right)^2\right]\right\}^{\theta-1} \tag{8}$$

for $x>0,\theta>0,\lambda>0$

where; θ is a shape parameter

λ is a scale parameter

The shape of the Burr X-Exponential distribution could be unimodal (for instance, when $"\theta=2,\lambda=3","\theta=2,\lambda=0.3"$) or decreasing (for instance, when $"\theta=0.3,\lambda=0.5","\theta=0.7,\lambda=2"$).

Reliability Analysis

Here, the survival function, hazard function, odds function and reversed hazard function for the Burr X-Exponential distribution are derived.

Survival Function

The mathematical expression for survival function is;

$$S(x)=1-F(x) \tag{9}$$

Where; $F(x)$ is as defined in Equation (7).

Therefore, the expression for the survival function of the Burr X-Exponential distribution is:

$$S(x)=1-\left\{1-\exp\left[-\left(\frac{1-\exp(-\lambda x)}{\exp(-\lambda x)}\right)^2\right]\right\}^\theta \tag{10}$$

for $x>0,\theta>0,\lambda>0$

Hazard Function

The mathematical expression for hazard function is:

$$h(x)=\frac{f(x)}{1-F(x)} \tag{11}$$

where; $F(x)$ and $f(x)$ are as defined in Equations (7) and (8) respectively.

Therefore;

$$h(x)=\frac{\frac{2\theta\lambda\left[1-e^{(-\lambda x)}\right]}{\left(e^{(-\lambda x)}\right)^2}\exp\left[-\left(\frac{1-e^{(-\lambda x)}}{e^{(-\lambda x)}}\right)^2\right]\left\{1-\exp\left[-\left(\frac{1-e^{(-\lambda x)}}{e^{(-\lambda x)}}\right)^2\right]\right\}^{\theta-1}}{1-\left\{1-\exp\left[-\left(\frac{1-e^{(-\lambda x)}}{e^{(-\lambda x)}}\right)^2\right]\right\}^\theta} \tag{12}$$

for $x>0,\theta>0,\lambda>0$

It is interesting to note that the plots at various parameter values indicate that the shape of the hazard function of the Burr X-Exponential distribution is increasing.

Odds Function

Odds function is mathematically defined by:

$$O(x)=\frac{F(x)}{S(x)} \tag{13}$$

Therefore, the odds function for the Burr X-Exponential distribution is:

$$O(x)=\frac{\left\{1-\exp\left[-\left(\frac{1-\exp(-\lambda x)}{\exp(-\lambda x)}\right)^2\right]\right\}^\theta}{1-\left\{1-\exp\left[-\left(\frac{1-\exp(-\lambda x)}{\exp(-\lambda x)}\right)^2\right]\right\}^\theta} \tag{14}$$

for $x>0,\theta>0,\lambda>0$

Reversed Hazard Function

Reversed hazard function is given by:

$$r(x)=\frac{f(x)}{F(x)} \tag{15}$$

Therefore, the expression for the reversed hazard function of the Burr X-Exponential distribution is:

$$r(x)=\frac{\frac{2\theta\lambda\left[1-\exp(-\lambda x)\right]}{\left(\exp(-\lambda x)\right)^2}\exp\left[-\left(\frac{1-\exp(-\lambda x)}{\exp(-\lambda x)}\right)^2\right]}{\left\{1-\exp\left[-\left(\frac{1-\exp(-\lambda x)}{\exp(-\lambda x)}\right)^2\right]\right\}} \tag{16}$$

for $x>0,\theta>0,\lambda>0$

Quantile Function and Median

The quantile function is derived from;

$$Q(u) = F^{-1}(u) \quad (17)$$

Therefore, the quantile function for the Burr X-Exponential distribution is given by;

$$Q(u) = -\lambda^{-1}\log\left[\frac{1}{\left[-\log\left(1-u^{1/\theta}\right)\right]^{1/2}+1}\right] \quad (18)$$

where; $u \sim Uniform(0,1)$.

That means, random samples can be generated from the Burr X-Exponential distribution using:

$$X = -\lambda^{-1}\log\left[\frac{1}{\left[-\log\left(1-u^{1/\theta}\right)\right]^{1/2}+1}\right] \quad (20)$$

where; $u \sim Uniform(0,1)$.

Estimation of Parameters

Let $x_1, x_2, \ldots, x_n$ denote random samples from the Burr X-Exponential distribution with parameters θ and λ, using the method of maximum likelihood estimation (MLE), the likelihood function is given by:

$$f(x_1, x_2, \ldots, x_n; \theta, \lambda) = \prod_{i=1}^{n}\left[\frac{2\theta\lambda\left[1-\exp(-\lambda x_i)\right]}{\left(\exp(-\lambda x_i)\right)^2}\exp\left[-\left(\frac{1-\exp(-\lambda x_i)}{\exp(-\lambda x_i)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{1-\exp(-\lambda x_i)}{\exp(-\lambda x_i)}\right)^2\right]\right\}^{\theta-1}\right]$$

Let $l = \log f(x_1, x_2, \ldots, x_n; \theta, \lambda)$ denote the log-likelihood function, then:

$$l = n\log(2) + n\log(\theta) + n\log(\lambda) + \sum_{i=1}^{n}\log\left[1-\exp(-\lambda x_i)\right] + 2\lambda\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\left(\frac{1-\exp(-\lambda x_i)}{\exp(-\lambda x_i)}\right)^2 +$$

$$(\theta-1)\sum_{i=1}^{n}\log\left[1-\exp\left(-\left(\frac{1-\exp(-\lambda x_i)}{\exp(-\lambda x_i)}\right)^2\right)\right] \quad (21)$$

Differentiating Equation (21) with respect to parameters θ and λ, setting the resulting non-linear system of equations to zero and solving them simultaneously gives the maximum likelihood estimates of parameters θ and λ respectively. It is much easier to solve these equations using algorithms in statistical software like R and so on when data sets are available.

III APPLICATIONS TO REAL LIFE DATA

In this section, the Burr X-Exponential distribution and Exponential distribution are applied to three real data. Here, judgment is based on the Log-likelihood and Akaike Information Criterion (AIC) values posed by these distributions.

Data I: It represents the height of 100 female athletes (measured in cm) collected at the Australian Institute of Sport. The data has previously been used by Cook and Weisberg (1994), Al-Aqtash et al., (2014) and Owoloko et al., (2016).

The summary of Data I is shown in Table 1:

Table 1: Summary of Data on Height of 100 Female Athletes

N	Mean	Variance	Skewness	Kurtosis
100	174.6	67.9339	-0.5598	4.1967

The performance of the Burr X-Exponential distribution with respect to its baseline distribution is as shown in Table 2:

Table 2: Burr X-Exponential distribution Versus Exponential distribution (with standard error in parentheses)

Distributions	Estimates	Log-Likelihood	AIC
Burr X-Exponential	$\hat{\theta} = 1.563e-01$ $(1.696e-02)$ $\hat{\lambda} = 2.292e-04$ $(3.041e-05)$	-747.0396	1498.079
Exponential	$\hat{\lambda} = 0.0057276$ (0.0005729)	-616.2463	1234.493

DATA II: The second data is from an accelerated life test of 59 conductors. The data has previously been used by Nasiri et al., (2011) and Oguntunde et al., (2016). The observations are as follow:

The data summary is as shown in Table 3:

Table 3: Summary of data on accelerated life test of conductors

N	Mean	Variance	Skewness	Kurtosis
59	6.929	2.4801	0.2196	3.2809

The performance of the Burr X-Exponential distribution is assessed using Data II and the result is as shown in Table 4:

Table 4: Performance of Burr X-Exponential distribution using Data II (with standard error in parentheses)

Distributions	Estimates	Log-Likelihood	AIC
Burr X-Exponential	$\hat{\theta}=0.171916$ (0.024484) $\hat{\lambda}=0.007877$ (0.001289)	-243.8967	491.7933
Exponential	$\hat{\lambda}=0.14432$ (0.01879)	-173.2091	348.4182

Data III: This data represents the relief times (in minutes) of patients receiving an analgesic. The data has been used recently by Shanker et al., (2015) to assess the flexibility of Exponential distribution and Lindley distribution. The observations are as follow:

The data summary is as shown in Table 5:

Table 5: Summary of data on patients receiving analgesic

N	Mean	Variance	Skewness	Kurtosis
20	1.900	0.4957895	1.71975	5.924108

The performance of the Burr X-Exponential distribution is assessed using Data III and the result is as shown in Table 6:

Table 6: Performance of Burr X-Exponential distribution using Data III (with standard error in parentheses)

Distributions	Estimates	Log-Likelihood	AIC
Burr X-Exponential	$\hat{\theta}=0.22180$ (0.05557) $\hat{\lambda}=0.05540$ (0.01350)	-51.85801	107.716
Exponential	$\hat{\lambda}=0.5263$ (0.1177)	-32.83708	67.67416

Remarks: The lower the AIC value, the better the model and the higher the log-likelihood value the better the model.

IV Conclusion

The Burr X-Exponential distribution has been successfully developed, its statistical properties like the quantile function, median, survival function, hazard function, reversed hazard function and odds function have been explicitly established. The shape of the distribution has been investigated to be either unimodal or decreasing (depending on the parameter values). The model has been applied to three different data and its performance was compared to the baseline distribution (Exponential distribution), it is evident from the analysis that the Burr X-Exponential distribution failed to perform better than the baseline distribution in all the applications provided based on the AIC and log-likelihood values posed by these distributions; this result contradicts that of Yousof et al., (2016) where the Burr X-Lomax distribution performed better than the Lomax distribution (its baseline distribution).

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REFERENCES

[1] R. Al-Aqtash, C. Lee and F. Famoye. "Gumbel-Weibull Distribution: Properties and Applications", *Journal of Modern Applied Statistical Methods*, 13, 201-225, 2014.

[2] A. Alzaatreh, F. Famoye and C. Lee. "Weibull-Pareto Distribution and Its Applications", *Communications in Statistics-Theory and Methods*, 42 (9), 1673-1691, 2013.

[3] M. Bourguignon, R. B. Silva and G. M. "Cordeiro. The Weibull-G Family of Probability Distributions", *Journal of Data Science*, 12, 53-68, 2014.

[4] R. D. Cook and S. Weisberg. *An Introduction to Regression Graphics (First Edition)*, John Wiley and Sons, New York, ISBN-10: 0470317701, pp:280, 1994.

[5] G. M. Cordeiro and M. de Castro. "A New family of Generalized Distributions", *Journal of Statistical computation and Simulation*, 81, 883-898, 2011

[6] N. Eugene, C. Lee and F. Famoye. "Beta-Normal distribution and Its Applications", *Communications in Statistics: Theory and Methods*, 31, 497-512, 2002

[7] A. W. Marshall and I. Olkin. "A new method for adding a parameter to a family of distributions with application to the exponential and weibull families", *Biometrika*, 84 (3), 641-652, 1997.

[8] P. Nasiri, I. Makhdoom and B. Yaghoubian. "Estimation Parameters of the Weighted Exponential Distribution", *Australian Journal of Basic and Applied Sciences*, 5(9), 2007-2014, 2011.

[9] P. E. Oguntunde, E. A. Owoloko and O. S. Balogun. "On A New Weighted Exponential Distribution: Theory and Application", *Asian Journal of applied Sciences*, 9(1), 1-12, 2016

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[10] E. A. Owoloko, P. E. Oguntunde and A. O. Adejumo. “A Comparative Analysis on the Performance of the Convoluted Exponential Distribution and the Exponential Distribution in terms of Flexibility”, *Journal of Mathematics and Statistics*, 12(1), 59-64, 2016
[11] W. Shaw and I. Buckley. “The alchemy of probability distributions: Beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map”. *Research Report.* 2007
[12] M. H. Tahir, G. M. Cordeiro, A. Alzaatreh, M. Mansoor and M. Zubair M. The Logistic-X Family of Distributions and Its Applications, *Communication in Statistics-Theory and Methods* (To Appear), 2016
[13] H. M. Yousof, A. Z. Afify, G. G. Hamedani and G. Aryal G. The Burr X Generator of Distributions for Lifetime Data, 2016, Retrieved from: *https://www.researchgate.net/profile/Ahmed_Afify10/publication/305469745_The_Burr_X_GenGenera_of_Distributions_for_Lifetime_Data/links/5790bf6608ae108aa04017f2.pdf*