

Variational Iteration Method for Natural Frequencies of a Cantilever Beam with Special Attention to the Higher Modes

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Abstract— In this work, the variational iteration method (VIM) is used to calculate the natural frequencies of a cantilever prismatic beam especially for the higher modes of vibration. The solutions yielded by VIM are validated by comparing with the natural frequencies of the said beam for lower modes earlier obtained using analytical method and the differential transform method.

Index Terms— cantilever beam, natural frequency, prismatic beam, variational iteration method

I. INTRODUCTION

SEVERAL techniques have been used to carry out the vibration analysis of beams with a view to determining their vibration characteristics. Lai, et al. [1] analysed the free vibration of uniform Euler-Bernoulli beam with different elastically supported conditions using Adomian decomposition method (ADM). Li [2] had earlier studied the vibration characteristics of a beam having general boundary conditions. The displacement of the beam was sought in form of a linear combination of a Fourier series and an auxiliary polynomial function. Kim and Kim [3] also applied Fourier series to determine the natural frequencies of beams having generally restrained boundary conditions. Later, Liu and Gurram [4] adopted the He's variational iteration method to estimate the vibration frequencies of a uniform Euler-Bernoulli beam for various supporting end conditions. Natural frequencies for the first six modes of vibration were presented in their work.

Malik and Dang [5] employed the differential transform method (DTM) to obtain the natural frequencies and mode shapes of a uniform Euler-Bernoulli beam. The natural frequencies and the mode shape functions for the first three

modes of vibration were derived for all the possible combinations of the classical end conditions of a beam. The primary aim of this present work is to, therefore, explore further the versatility of the VIM in determining the vibration characteristics of a uniform cantilever beam especially for higher modes.

II. PROBLEM FORMULATION

The governing equation of motion for free vibration of a uniform Euler-Bernoulli beam is given as:

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (1)$$

where $y(x,t)$ is the transverse displacement of the beam at any distance x along the length of the beam at time t . In Eq. (1), E , and ρ are Young's modulus of elasticity and mass density of the beam material respectively while A and I are the cross-sectional area and the area moment of inertia of the cross section of the beam, respectively.

The boundary conditions for a cantilever beam can be written as:

$$y(0,t) = 0, \quad \frac{\partial y(0,t)}{\partial x} = 0, \quad (2)$$

and

$$\frac{\partial^2 y(L,t)}{\partial x^2} = 0, \quad \frac{\partial^3 y(L,t)}{\partial x^3} = 0, \quad (4)$$

where L is the length of the beam.

For Eqs. (1) and (2), we assume a solution of the form:

$$y(x,t) = Y(x)(\cos \omega t + \theta), \quad (5)$$

where $Y(x)$ is the mode shape of the beam and ω is the angular frequency of the system.

By using Eq. (5) in Eqs. (1) and (2) and after carrying out non-dimensionalization, the equations of motion reduce to

Manuscript received March 04, 2017; revised March 20, 2017. This work was supported fully by Covenant University, Ota, Nigeria.

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$$\frac{d^4 Y(\xi)}{d\xi^4} - \Omega^2 Y(\xi) = 0, 0 < \xi < 1, \quad (6)$$

such that

$$\xi = \frac{x}{L}, Y(\xi) = \frac{Y(x)}{L}, \Omega^2 = \frac{\rho A \omega^2 L^4}{EI}, \quad (7)$$

where Ω is the nondimensional natural frequency of the cantilever beam.

Also, the boundary conditions can be written in non-dimensional form as

$$Y(0) = 0, \frac{dY(0)}{d\xi} = 0, \quad (8)$$

and

$$\frac{d^2 Y(1)}{d\xi^2} = 0, \frac{d^3 Y(1)}{d\xi^3} = 0. \quad (9)$$

III. OUTLINE OF THE METHOD

The basic principle of variational iteration method is presented here. Consider the differential equation of the form

$$Lw(t) + Nw(t) = g(t), \quad (10)$$

where L and N are respectively linear and nonlinear operators, $g(t)$ is a known function referred to as the nonhomogeneous term. A correction functional presented by the variational iteration method for Eq. (10) is given by

$$w_{n+1}(\xi) = w_n(\xi) + \int_0^\xi \lambda(\tau) (Lw_n(\tau) + N\tilde{w}_n(\tau)) d\tau, \quad (11)$$

where λ is a general Lagrange multiplier, which can be optimally determined by variational theory, \tilde{w}_n is a restricted variation which implies that $\delta \tilde{w}_n = 0$, where δ is the variational derivative.

IV. SOLUTION BY VIM

The correction functional for the governing equation of the vibration problem can be written as

$$Y_{n+1}(\xi) = Y_n(\xi) + \int_0^\xi \lambda(\tau) \left(\frac{d^4 Y_n(\tau)}{d\tau^4} - \Omega^2 Y_n(\tau) \right) d\tau, \quad (12)$$

which is obtained by comparing Eq. (6) with Eq. (10).

Using the method of integration by parts, Eq. (12) can be written as

$$\begin{aligned} Y_{n+1}(\xi) = & Y_n(\xi) + \lambda(\xi) \frac{d^3 Y_n(\xi)}{d\xi^3} - \frac{d\lambda(\xi)}{d\xi} \frac{d^2 Y_n(\xi)}{d\xi^2} \\ & + \frac{d^2 \lambda(\xi)}{d\xi^2} \frac{dY_n(\xi)}{d\xi} - \frac{d^3 \lambda(\xi)}{d\xi^3} Y_n(\xi) \\ & + \int_0^\xi \left[\frac{d^4 \lambda(\tau)}{d\tau^4} - \Omega^2 \lambda(\tau) \right] Y_n(\tau) d\tau \end{aligned} \quad (13)$$

Taking the variation of Eq. (13) with respect to Y_n gives

$$\begin{aligned} \delta Y_{n+1}(\xi) = & \delta Y_n(\xi) + \lambda(\xi) \delta \frac{d^3 Y_n(\xi)}{d\xi^3} \\ & - \frac{d\lambda(\xi)}{d\xi} \delta \frac{d^2 Y_n(\xi)}{d\xi^2} + \frac{d^2 \lambda(\xi)}{d\xi^2} \delta \frac{dY_n(\xi)}{d\xi} \\ & - \frac{d^3 \lambda(\xi)}{d\xi^3} \delta Y_n(\xi) \\ & + \int_0^\xi \left[\frac{d^4 \lambda(\tau)}{d\tau^4} - \Omega^2 \lambda(\tau) \right] \delta Y_n(\tau) d\tau \end{aligned} \quad (14)$$

It can be readily shown that the Lagrange multiplier is

$$\lambda(\tau) = \frac{(\tau - \xi)^3}{3!}. \quad (15)$$

(Wazwaz [6]).

Substituting this value of the Lagrange multiplier in Eq. (15) into the correction functional stated in Eq. (12), the following iteration formula can be obtained:

$$Y_{n+1}(\xi) = Y_n(\xi) + \int_0^\xi \frac{(\tau - \xi)^3}{3!} \left(\frac{d^4 Y_n(\tau)}{d\tau^4} - \Omega^2 Y_n(\tau) \right) d\tau. \quad (16)$$

For fast convergence, the function $Y_0(\xi)$ is required to be selected by using the initial conditions as follows:

$$Y_0(\xi) = Y(0) + \frac{dY(0)}{d\xi} \xi + \frac{1}{2!} \frac{d^2 Y(0)}{d\xi^2} \xi^2 + \frac{1}{3!} \frac{d^3 Y(0)}{d\xi^3} \xi^3 \quad (17)$$

for fourth order $\frac{d^4 Y_n}{d\xi^4}$.

Several approximations as listed below can be obtained from Eq. (16):

$$\begin{aligned} Y_1(\xi) = & Y_0(\xi) + \int_0^\xi \frac{(\tau - \xi)^3}{3!} \left(\frac{d^4 Y_0(\tau)}{d\tau^4} - \Omega^2 Y_0(\tau) \right) d\tau \\ Y_2(\xi) = & Y_1(\xi) + \int_0^\xi \frac{(\tau - \xi)^3}{3!} \left(\frac{d^4 Y_1(\tau)}{d\tau^4} - \Omega^2 Y_1(\tau) \right) d\tau \\ & \vdots \\ Y_k(\xi) = & Y_{k-1}(\xi) + \int_0^\xi \frac{(\tau - \xi)^3}{3!} \left(\frac{d^4 Y_{k-1}(\tau)}{d\tau^4} - \Omega^2 Y_{k-1}(\tau) \right) d\tau \end{aligned} \quad (18)$$

Thus, the solution to Eq. (6) is obtained as the limit of the above resulting successive approximations expressed by

$$Y(\xi) = \lim_{k \rightarrow \infty} Y_k(\xi). \quad (19)$$

In practice, a large number, say s , which is to be determined based on the required accuracy, is chosen to replace infinity in Eq. (20). So, one has

$$Y(\xi) = Y_s(\xi). \quad (20)$$

Now, using Eq. (8) in Eq. (17), we get

$$Y_0(\xi) = \frac{1}{2!} \frac{d^2 Y(0)}{d\xi^2} \xi^2 + \frac{1}{3!} \frac{d^3 Y(0)}{d\xi^3} \xi^3, \quad (21)$$

or

$$Y_0(\xi) = \frac{1}{2} a_1 \xi^2 + \frac{1}{6} a_2 \xi^3, \quad (22)$$

where

$$a_1 = \frac{d^2 Y(0)}{d\xi^2}, \quad a_2 = \frac{d^3 Y(0)}{d\xi^3}. \quad (23)$$

Substituting Eq. (22) into Eqs. (18) leads to the following successive approximations

$$Y_1(\xi) = \left(\frac{1}{2} \xi^2 + \frac{1}{48} \Omega^2 \xi^6 \right) a_1 + \left(\frac{1}{6} \xi^3 + \frac{1}{144} \Omega^2 \xi^7 \right) a_2, \quad (24)$$

$$Y_2(\xi) = \left(\frac{1}{2} \xi^2 + \frac{1}{24} \Omega^2 \xi^6 + \frac{1}{1152} \Omega^4 \xi^{10} \right) a_1 + \left(\frac{1}{6} \xi^3 + \frac{1}{72} \Omega^2 \xi^7 + \frac{1}{3456} \Omega^4 \xi^{11} \right) a_2, \quad (25)$$

$$Y_3(\xi) = \left(\frac{1}{2} \xi^2 + \frac{1}{16} \Omega^2 \xi^6 + \frac{1}{384} \Omega^4 \xi^{10} + \frac{1}{27648} \Omega^6 \xi^{14} \right) a_1 + \left(\frac{1}{6} \xi^3 + \frac{1}{48} \Omega^2 \xi^7 + \frac{1}{1152} \Omega^4 \xi^{11} + \frac{1}{82944} \Omega^6 \xi^{15} \right) a_2, \quad (26)$$

and so on.

Thus, it is seen that $Y_n(\xi)$ takes the form

$$Y_n(\xi) = a_1 f^{[n]}(\xi, \Omega) + a_2 g^{[n]}(\xi, \Omega), \quad (27)$$

where $f^{[n]}(\xi, \Omega)$ and $g^{[n]}(\xi, \Omega)$ are functions of ξ and Ω associated with n .

Let $Y_n(\xi) = Y(\xi)$, then Eqs. (9) become

$$a_1 \frac{d^2 f^{[n]}(1, \Omega)}{d\xi^2} + a_2 \frac{d^2 g^{[n]}(1, \Omega)}{d\xi^2} = 0, \quad (28)$$

and

$$a_1 \frac{d^3 f^{[n]}(1, \Omega)}{d\xi^3} + a_2 \frac{d^3 g^{[n]}(1, \Omega)}{d\xi^3} = 0. \quad (29)$$

Eqs. (28) and (29) can be put in the vector-matrix form

$$\begin{bmatrix} \frac{d^2 f^{[n]}(1, \Omega)}{d\xi^2} & \frac{d^2 g^{[n]}(1, \Omega)}{d\xi^2} \\ \frac{d^3 f^{[n]}(1, \Omega)}{d\xi^3} & \frac{d^3 g^{[n]}(1, \Omega)}{d\xi^3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (30)$$

The system of equations in (30) possesses nontrivial solutions provided the determinant of the coefficient matrix is equal to zero.

That is,

$$\begin{vmatrix} \frac{d^2 f^{[n]}(1, \Omega)}{d\xi^2} & \frac{d^2 g^{[n]}(1, \Omega)}{d\xi^2} \\ \frac{d^3 f^{[n]}(1, \Omega)}{d\xi^3} & \frac{d^3 g^{[n]}(1, \Omega)}{d\xi^3} \end{vmatrix} = 0. \quad (31)$$

The values of Ω yielded by solving Eq. (31) are the nondimensional natural frequencies of the beam under consideration. In essence, one can obtain the j th estimated nondimensional frequency, $\Omega_j^{[n]}$ corresponding to n . The value of n would be decided by this inequality:

$$|\Omega_j^{[n]} - \Omega_j^{[n-1]}| \leq \varepsilon \quad (32)$$

where ε is the allowable error tolerance dependent on the required accuracy.

V. NUMERICAL EXAMPLE

In this section, the procedures of variational iteration method for solving the vibration problem in this study earlier discussed are implemented numerically with a view to computing the nondimensional frequencies of the cantilever beam being studied. in this paper. The results obtained for the first six vibration modes are compared with the results obtained analytically and by differential transform method. These are displayed in Table I. It can be observed from Table I that there is an excellent agreement between the results obtained by DTM and VIM.

The nondimensional frequencies for higher modes of vibration which are obtained using VIM are reported in Table II.

TABLE I
COMPARISON OF THE FIRST SIX NONDIMENSIONAL NATRURAL
FREQUENCIES BY METHODS FOR CANTILEVER EULER-
BERNOULLI BEAM

Nondimensional Frequencies	Analytical method [7]	DTM [5]	VIM (Present)
Ω_1	3.51563	3.516015	3.516015
Ω_2	22.03364	22.034492	22.034492
Ω_3	61.7010	61.697214	61.697214
Ω_4	120.9120	-	120.901916
Ω_5	199.8548	-	199.859530
Ω_6	298.5638	-	298.555531

TABLE II
NONDIMENSIONAL NATRURAL FREQUENCIES FOR CANTILEVER
EULER-BERNOULLI BEAM FOR HIGHER MODES OF VIBRATION

Nondimensional Frequencies	VIM (Present)
Ω_7	416.990786
Ω_8	555.165247
Ω_9	713.078922
Ω_{10}	890.731667
Ω_{11}	1088.127450
Ω_{12}	1305.168455
Ω_{13}	1544.058323
Ω_{14}	1771.488309
Ω_{15}	2401.026455
Ω_{16}	4147.708537
Ω_{17}	8575.141626
Ω_{18}	14725.598220
Ω_{19}	21490.698220
Ω_{20}	29555.946880

VI. CONCLUSION

The variational iteration method (VIM) has been used to estimate the natural frequencies of a cantilever prismatic beam with emphasis on the natural frequencies for higher modes of vibration. The efficiency and the reliability of the method was also established by showing that there is an excellent agreement between VIM results and the existing results in the literature.

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