On the Application of Martingale theory to Investment Strategy

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Abstract - ***Most often than not, an investor holding stock must decide whether to sell or keep holding the stock. This investment strategy over the years appears to be an easy task to take. In the investment parlance, it is called the Broker's Common Sense (BCS). We have shown in this paper that the so-called BCS strategy is backed with advanced mathematical (probabilistic) phenomenon; we used the martingale theory to describe the strategy.***

Index words: Martingale theory, probability, investment strategy, up-crossing, probability measure.

I INTRODUCTION

In the literature, different approaches have been proposed on the best time for an investor to buy or sell shares or to buy and hold shares perpetually. A variational inequality for solving optimal stopping problem in deciding on when to sell a share (stock) was introduced in [1]. A risky asset in a financial market is usually described as a stochastic process on some probability space (Ω, f, P). In [2], a stock selling problem whose stock price is given by a geometric Brownian motion with constant coefficients was considered.

Different buying or selling strategies are usually adopted in the literature, see ([3], [4] and [5]). In an earlier work, [7], gave an empirical evidence from an emerging market on when to sell or hold a stock using optimal time approach.

The commonest approach usually is to buy when the price is low and sell when the price is high.

Our interest in this paper, is to examine if this approach is always appropriate; as nice as it may be seen. We shall adopt the martingale theory to investigate if the buy low and sell high strategy is indeed appropriate.

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II PRELIMENARIES

Let (Ω, f, P) be a probability space, where Ω is a set, f a $\sigma - a\lg ebra$ of subsets of Ω and P a probability measure defined on f.

Let I be any interval of the form, (a,b), $[a,b),(a,b]$ or $[a,b]$ of the ordered set $\{-\infty, ..., -1, 0, 1, ..., \infty\}$.

Let $\{f_n, n \in I\}$ be an increasing sequence of $\sigma -$ algebras of f sets. Suppose that $\{Z_n, n \in I\}$ is a sequence of random variables on Ω satisfying that

(i). Z_n is measurable with respect to f_n,

(ii). $E|Z_n| < \infty$,

(iii). $E(Z_n | f_m) = Z_m$ almost surely (a. s) for all $m < n,\ m, n \in I.$

Then the sequence $\{Z_n, n \in I\}$ is said to be a martingale with respect to $\{f_n, n \in I\}$.

If (i) and (ii) are retained and (iii) is replaced by the inequality, $E(Z_n | f_m) \geq Z_m\ a.s\ or\ \ (E(Z_n | f_m) \leq Z_m\ a.s)$ then $\{Z_n, n \in I\}$ is called a sub-martingale (supper martingale) with respect to $\{f_n, n \in I\}$.

III MATHEMATICAL FORMULATIONS

We looked at the simplest strategy in the market, which is to buy low and sell high. Let X_n be the price of a share at time $n.$ We assumed that the market is favourable, that is, we assumed that X_n is a sub-martingale; in other words we expect the investor to earn money in the long run. Finally, we considered the interval $[a,b]$, where the investor buys at level a and sell at level $b.$ The investor does not buy and sell immediately, he holds on to the share for some time. The period the investor holds on to the share is called an *up-crossing.*

Definition 1 A sequence of random variable (H_n) is called predictable, with respect to a filtration f_n, if H_n is $f_{n-1}-$ measurable.

Definition 2 A sequence $(X_k, X_{k+1}, ..., X_m)$ is an up-crossing if $X_k \le a \le X_{k+1} \le X_{k+2} \le ... \le X_{m-1} \le b \le X_m$.

The investor's strategy is as follows; when the price of the stock drops below a first time, buy a share. Hold onto it until the price of the stock increases above b first time; repeat this process. During every up-crossing U_n, the investor earns at least $(b-a)$ dollars per stock. Then the guaranteed profit is at least $(b-a)n.$

We shall invoke the following theorems:

Theorem 1 Let (X_n) be a super-martingale. Let (H_n) be a predictable sequence, and suppose each $H_n \ge 0$ is bounded. Then $(H_n.X_n) = (H.X)_n$ is a super-martingale.

Proof:

$$E\left((H.X)_{n+1} \mid f_n\right) = E\left((H.X)_n + H_{n+1}\left(X_{n+1} - X_n\right) \mid f_n\right)$$
$$= (H.X)_n + H_{n+1} E\left(X_{n+1} - X_n \mid f_n\right)$$
$$\le (H.X)_n.$$

Since $(H.X)_n$ and H_{n+1} are f_n-measurable, $H_{n+1} \ge 0$ and (X_n) a super-martingale given.

Theorem 2 Up-crossing Inequality. Let X_n be a sub-martingale, and U_n the number of up-crossings in the interval $[a,b]$ at time $n.$ Then

$$E(U_n) \le \frac{E(X_n - X_0)}{b-a} \quad (1)$$

Remark to theorem 2**:** If we multiply both sides of (1) by $(b-a)$, we obtain

$$E\left[(b-a)U_n\right] \le E\left(X_n - X_0\right) \quad (2)$$

The LHS of (2) is the profit using the strategy 'buy low, sell high', while the RHS of (2) is the profit obtained when the strategy 'buy and hold' and sell after time n regardless of X_n is used.

Proof of theorem 2**:** We realise the investor's strategy 'buy low, sell high' as a predictable sequence. After time k, what the investor is doing is

$$H_k = \begin{cases} 1 & \textit{if the investor holds a share} \\ 0 & \textit{otherwise} \end{cases}$$

The profit therefore is

$$(H.X)_n = \sum_{k=1}^{n} H_k\left(X_k - X_{k-1}\right) \quad (3)$$

When H =1, then the RHS of (3) is a telescoping sum, which gives $X_{last} - X_{first}$.

We know that the profit is bounded below:

$$(H.X)_n \ge (b-a)U_n \quad (4)$$

Now, we use theorem 1 and that (4) is a super martingale to look at the empty spaces, where we do not hold a share.

Define

$$H'_n = 1 - H_n \quad (5)$$

which indicate the times we do not hold share, easily, we have

$$(H.X)_n + (H'.X)_n = (H.X)_n + (1-H)_n = (\mathrm{H}.X)_n \quad (6)$$

The first term on the LHS of (6) is the profit in 'buy low, sell high'. The RHS of (6) is the profit in 'buy and hold'. There is the need to analyse the difference given by $(H'.X)_n$.

Recall that (X_n) is a sub-martingale, and (H'_n) is predictable. By the dual formulation of our theorem, $(H'.X)_n$ is a sub-martingale.

Thus, $E(H'.X)_n \ge E(H'.X)_0 = 0 \quad (7)$

We apply expectation to (6) and use (7) to obtain

$$E(H.X)_n \le E\left(X_n - X_0\right) \quad (8).$$

Combining (8) with (4) completes the proof.

The implication of (8) is that the martingale is either bounded in the end or it is not. If it is not bounded, then it will not respect the midpoint of $[a,b].$ In that case, the number of crossing will be small. An application of this approach, though with some modifications can be found in [6] and [8].

IV Conclusion

We have shown in this paper how martingale technique can be applied to solving the broker's strategy. This short note has shown that what we might term 'common sense' in the investment world is not as common as it may appear but are backed up with advanced mathematical (probabilistic) phenomenon.

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