Dynamic Modeling and Analysis of Inverted Pendulum using Lagrangian-Differential Transform Method

Michael C. Agarana, *Member, IAENG,* and Oluseyi O. Ajayi

Abstract— **There has been an increase in application of pendulum in robotics which is applicable in Medicine, Agriculture, Military, Industries Explorations and Entertainment. Many Researchers have shown interest in inverted pendulum stability and control in recent years. This study is concerned with the development of equation of motion of inverted pendulum and its analysis by adopting the Lagrangian and Differential Transform Method (DTM) respectively. The resulting equations were solved analytically. Results obtained are consistent with the ones in the literature.**

Index Terms—Applied Mechanics, Dynamics Differential transform method. Inverted pendulum, Lagrangian.

I. INTRODUCTION

AN ordinary pendulum is one with the pivot at the top and the mass at the bottom. An inverted pendulum is the opposite way round. The pivot is at the bottom and the mass is on top. The inverted pendulum is a mechanism for carrying an object form one place to another and this is how it functions during walking. The "passenger unit" as Perry would call it is carried forward by the outstretched leg as it pivots over the foot [1]. So an inverted pendulum is a pendulum that has its centre of mass above its pivot point. It is often implemented with the pivot point mounted on a moving body that can move horizontally. Unlike a normal pendulum which is always stable when hanging downwards, an inverted pendulum is inherently unstable, and must be actively balanced in order to remain upright. This can be done in different ways including the pivot point horizontally as part of a feedback system [2,3]. A simple demonstration of this is achieved by balancing an upturned broomstick on the end of

one's finger. The inverted pendulum is a classic problem in dynamics and control theory and is used as a benchmark for testing control strategies[3,4]. Differential transform method is a semi analytical method which requires no discretization [4,5]. One of the advantages of DTM over others is that it is simple to apply [5,6,7]. The solution of the governing equation is very close to the analytical solutions [8]. The Lagrangian is a mathematical function of the generalized coordinates, their time derivatives, and time. It contains information about the dynamics of the system. Lagrangian mechanics is ideal for systems with conservative forces and for bypassing constraint forces in any coordinate system. Dissipative and driven forces can be accounted for by splitting the external forces into a sum of potential and non-potential forces [9]. When Newton's formulation of classical mechanics is not convenient, Lagrangian mechanics is widely used to solve mechanical problems in physics and engineering. Lagrangian mechanics applies to the dynamics of particles, and used in optimisation problems of dynamic systems. Lagrangian mechanics uses the energies (both kinetic and potential) in the system instead of forces. Thus a function that summarizes the dynamics of an entire system is simply known as the Lagrangian. Moreover, the Kinetic energy reflects the energy of the system's motion. The potential energy of the system reflects the energy of interaction between the particles, i.e. how much energy any one particle will have due to all the others and other external influences.[9]

The purpose of this study was to use differential transformation algorithm to solve the ordinary second order differential equation resulted from the deployment of Lagrangian to model the Kinematics of motion of inverted pendulum. DTM was then applied to solve the model so formed.[9] The dynamic effects of the pendulum through its angular displacement, angular velocity and

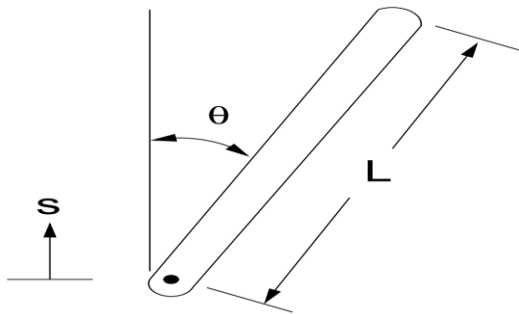angular acceleration were estimated over a period of time.

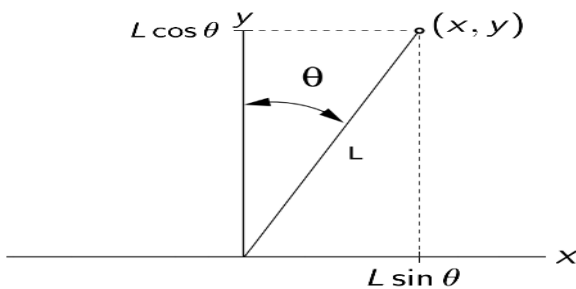Fig. 1 The angular displacement of the inverted pendulum

Fig. 2 The vertical and horizontal components of the length of the inverted pendulum

Fig. 1 represents an inverted pendulum, while Fig. 2 is the schematic of the inverted pendulum, showing the vertical and horizontal components of the length of the inverted pendulum. θ represents the angle the inverted pendulum makes with the vertical.

II. EQUATION OF MOTION

Using Lagrange's equations, which employ a single scalar function rather than the vector components, to derive the equations modeling an inverted pendulum, we take partial derivatives. In classical mechanics, the natural form of the Lagrangian is defined as

$$L = E_k - E_p \qquad (1)$$

Where E_K and E_P are kinetic and potential energies are respectively. E_p is defined by its mass m, and the gravitational constant g:

$$E_p = mgh \qquad (2)$$

The kinetic energy Ek of a point object is defined by its mass m and velocity v:

$$E_k = 1/2mv^2 \qquad (3)$$

Equation of motion can be directly derived by substitution using Euler Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad (4)$$

where θ is the angle the pendulum makes with the upward vertical.

Equation (2) can be written as
$$E_p = mgy \qquad (5)$$

Where h = y, is the height along vertical axis
From the figures 2a and 2b above,

$$x = L\sin \theta \qquad (6)$$
$$y = L\cos \theta \qquad (7)$$

This implies

$$\frac{dx}{dt} = L\cos (\theta) \frac{d\theta}{dt} \qquad (8)$$

and

$$\frac{dy}{dt} = -L\sin (\theta) \frac{d\theta}{dt} \qquad (9)$$

From equation (1), the Lagrangian can be written as

$$L = \frac{1}{2}mv^2 - mgy \qquad (10)$$

where h = y
Velocity as a vector quantity is given as

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \qquad (11)$$

Substituting equations (6) and (7) into equation (11) gives:

$$v^2 = L^2\left(\frac{d\theta}{dt}\right)^2\cos^2\theta + L^2\left(\frac{d\theta}{dt}\right)^2\sin^2\theta \qquad (12)$$

$$v^2 = L^2 \left(\frac{d\theta}{dt}\right)^2 \left[\cos^2\theta + \sin^2\theta\right] \qquad (13)$$

$$v^2 = L^2 \left(\frac{d\theta}{dt}\right)^2 \qquad (14)$$

Now substituting equations (7) and (14) into equation (10) gives:

$$L = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 - mgL\cos\theta \qquad (15)$$

which implies

$$\frac{\partial L}{\partial \theta} = mgL\sin\theta \qquad (16)$$

$$\frac{\partial L}{\partial(\frac{d\theta}{dt})} = mL^2\frac{d\theta}{dt} \qquad (17)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(\frac{d\theta}{dt})}\right) = mL^2\frac{d^2\theta}{dt^2} \qquad (18)$$

Substituting equation (16) and (18) into equation (4) gives:

$$mL^2\frac{d^2\theta}{dt^2} = mgL\sin\theta \qquad (19)$$

Therefore

$$\frac{d^2\theta}{dt^2} = \frac{g}{L}\sin\theta \qquad (20)$$

Without oscillator and taking g =10 and L =5 gives:

$$\ddot{\theta}(t) = 2\sin\theta \qquad (21)$$

subject to

$$\theta(0) = 0,\ \theta'(0) = -2 \qquad (22)$$

III MODEL SOLUTION

To solve equation (20), θ is solved for using Differential transform method (DTM). However equation (20) is a linear second order differential equation solvable by applying differential transform method.

A. Differential Transformation Method (DTM)

Differential transform method is a numerical method based on Taylor expansion. This method tries to find coefficients of series expansion of unknown function by using the initial data on the problem. The concept of differential transform method was first proposed by Zhou [10,11].

B. Definitions and Properties of DTM

With reference to article [4, 5, 6] the differential transformation of a function f(x) can be written as:

$$F(t) = \frac{1}{K!}\left[\frac{d^k f(x)}{dx}\right]x = 0 \qquad (23)$$

Where f(x) is referred to as the original function and F(x), the transformed function. The differential inverse of F(t) is

$$f(x) = \sum_{k=0}^{\infty} F(t)x^k \qquad (24)$$

Equation (26) is the Tailor series expansion of f(x) about x=0. It can also be written as

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k f(x)}{dx}\right] \qquad (25)$$

Some operational properties of DTM are as follows:

(i) If f(x) = p(x) $\pm$ g(x), then F(t) = p(t) $\pm$ F(t) $\qquad (26)$

(ii) If f(x) = cp(x), then F(t) = cP(t), where c is a constant. $\qquad (27)$

(iii) If f(x) = $\frac{d^n p(x)}{dx^n}$, then F(t) = $\frac{(k+n)!}{k}p(k+n)$ $\quad (28)$

(iv) If f(x) = sin(ax + b), then F(x) = $\frac{a^k}{k!}\sin\left(\frac{k\pi}{2}+b\right)$ (29)

(v) f(x) = $e^{\lambda x}$, then $F(t) = \frac{\lambda^k}{k!}$, where λ is a constant $\quad (30)$

C. Differential Transform Method on Second Order ODE)

Second order linear differential equations can be used as follows [4]

$$f(x) = \frac{1}{a}[r(x) - cy(x) - by^2 x] \qquad (31)$$

Applying the above properties, equation (31) becomes: [4]

$$F(k+2) = \frac{1}{a(k+1)(k+2)}[R(x) - b(k+1)F(k+1) - cF(k)]$$

$$(32)$$

which is subject to the initial condition that:

$$F(0) = F_0 = a_1 \text{ and } F^1(0) = F(1) = a_2 \qquad (33)$$

Equation (33) is called recursive formula.

Using equation (33) on equation (32) gives the appropriate solution.[4]

IV. APPLICATION OF DTM

To apply the discussed DTM to equation (20), equations (32) and (33) are re-written as:

$$DT[\overset{\square}{\theta}(t) = 2\sin\theta] \qquad (34)$$

$$\Rightarrow (k+1)(k+2)Y(k+2) = 2\sin Y(k) \qquad (35)$$

$$\therefore Y(k+2) = \frac{1}{(k+1)(k+2)}[2\sin Y(k)] \qquad (36)$$

with the initial conditions that $Y(0) = 0$, $Y'(0) = -2$.

$$(37)$$

Applying the recursive relation in (33) with $k \geq 0$ the values for Y(2), Y(3), Y(4), …, are obtained [4,6]:

For k=0:

$$Y(2) = \frac{1}{2}(0) = 0 \qquad (38)$$

For k=1:

$$Y(3) = \frac{1}{6}[2\sin(-2)] = -0.01163 \qquad (39)$$

For K =2:

$$Y(4) = \frac{1}{12}[2\sin 0] = 0 \qquad (40)$$

For K =3:

$$Y(5) = \frac{1}{20}[2\sin(-0.01163)] = -0.00002 \qquad (41)$$

For K =4:

$$Y(6) = \frac{1}{30}[2\sin 0] = 0 \qquad (42)$$

For k=5

$$Y(7) = \frac{1}{42}[2\sin(-0.00002)] = -0.00000002 \qquad (43)$$

And so on.

Generally speaking, n = k+2, where k=0,1,2,3,…

Therefore,

Y(n) = 0, -0.01163, 0, -0.00002, 0, -0.00000002 …

From equation (24):

$$f(x) = \sum_{k=0}^{\infty} F(t)x^k \qquad (44)$$

$$y(x) = \sum_{k=0}^{\infty} Y(t)x^k \qquad (45)$$

$$\therefore y = y(0) + y(1)x + y(2)x^2 + y(3)x^3 + y(4)x^4$$
$$+ y(5)x^5 + y(6)x^6 + y(7)x^7 + ... \qquad (46)$$

$$= 0 - 2x + 0 - 0.01163x^3 + 0 - 0.00002x^5$$
$$+ 0 - 0.00000002x^7 + ... \qquad (47)$$

$$= -2x - 0.01163x^3 - 0.00002x^5$$
$$- 0.00000002x^7 - .... \qquad (48)$$

Equation (34) can now be written as:

$$\theta(t) = -2t - 0.01163t^3 - 0.00002t^5$$
$$- 0.00000002t^7 - ... \qquad (49)$$

For t = 1,2,3,4,5,…

Truncating the value of the angular displacement after the fourth term:

$$\theta(0) = 0 \qquad (50)$$

$$\theta(1) = -2 - 0.01163 - 0.00002$$
$$- 0.00000002 = -2.01165002 \qquad (51)$$

$$\theta(2) = -4 - 0.09304 - 0.00064$$
$$- 0.00000256 = -4.09368256 \qquad (52)$$

$$\theta(3) = -6 - 0.31401 - 0.00486$$
$$- 0.00004374 = -6.31891374 \qquad (53)$$

$$\theta(4) = -8 - 0.74432 - 0.02048$$
$$-0.0003278 = -8.7651278 \qquad (54)$$

$$\theta(5) = -10 - 1.45375 - 0.0625$$
$$-0.0015625 = -11.5178125 \qquad (55)$$

And so on.

For the velocity:

Differentiating equation (49), with respect to t gives

$$\dot{\theta}(t) = -2 - 0.03489t^2 - 0.0001t^4 - 0.00000014t^6 - ... \quad (56)$$

Truncating the value of the angular velocity

after the fourth term, gives $\qquad (57)$

$$\therefore \dot{\theta}(0) = -2 - 0 - 0 - 0 - ... = -2 \qquad (58)$$

$$\dot{\theta}(1) = -2 - 0.03489 - 0.0001$$
$$-0.00000014 = -2.03499014 \qquad (59)$$

$$\dot{\theta}(2) = -2 - 0.13956 - 0.0016$$
$$-0.00000896 = -2.14116896 \qquad (60)$$

$$\dot{\theta}(3) = -2 - 0.31401 - 0.0081$$
$$-0.00010206 = -2.32221206 \qquad (61)$$

$$\dot{\theta}(4) = -2 - 0.55824 - 0.0256$$
$$-0.00057344 = -2.5304565 \qquad (62)$$

$$\dot{\theta}(5) = -2 - 0.87225 - 0.0625$$
$$-0.0021875 = -2.9369375 \qquad (63)$$

And so on.

For the Acceleration:

Differentiating equation (56), with respect to t gives:

$$\ddot{\theta}(t) = -0.06978t - 0.0004t^3$$
$$-0.00000084t^5 - \qquad (64)$$

Truncating the value of the angular acceleration
after the fourth term, gives

$$\therefore \ddot{\theta}(0) = 0 \qquad (65)$$

$$\ddot{\theta}(1) = -0.06978 - 0.0004 - 0.00000084$$
$$= -0.07018084 \qquad (66)$$

$$\ddot{\theta}(2) = -0.13956 - 0.0032 - 0.00002688$$
$$= -0.14278688 \qquad (67)$$

$$\ddot{\theta}(3) = -0.20934 - 0.0108 - 0.00020412$$
$$= -0.22034 \qquad (68)$$

$$\ddot{\theta}(4) = -0.27912 - 0.0256 - 0.00086016$$
$$= -0016 \qquad (69)$$

$$\ddot{\theta}(5) = -0.3489 - 0.05 - 0.002625 = -0.5 \quad (70)$$
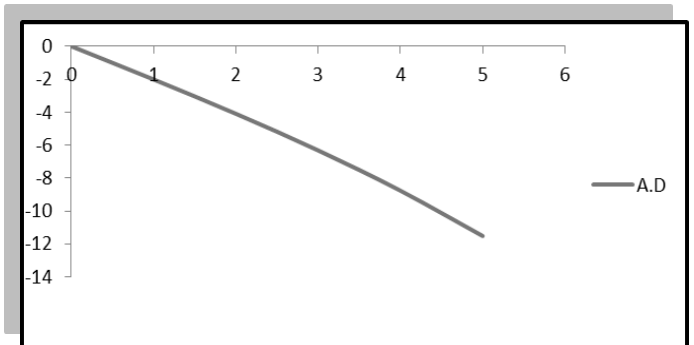
And so on.

Fig. 3: Angular Displacement of the pendulum as time increases
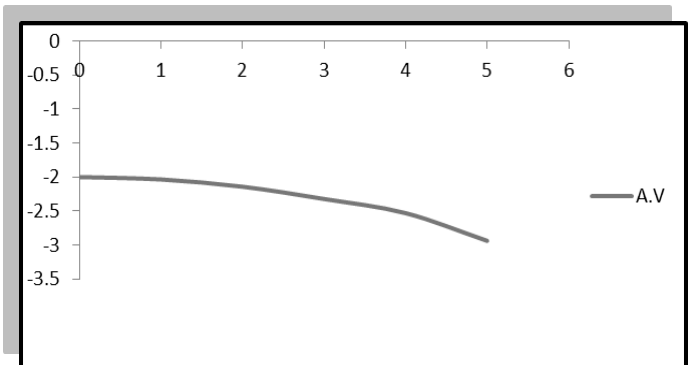

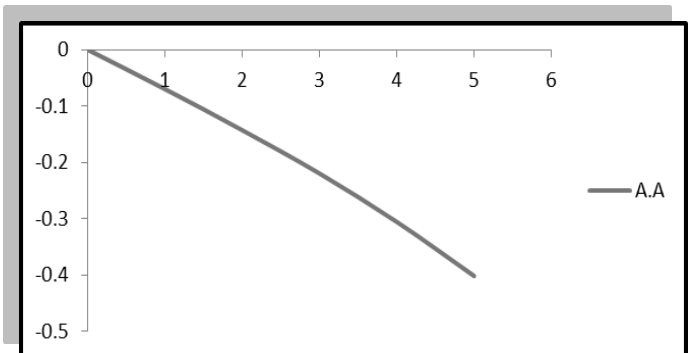Fig. 4: Angular velocity of the pendulum as time increases

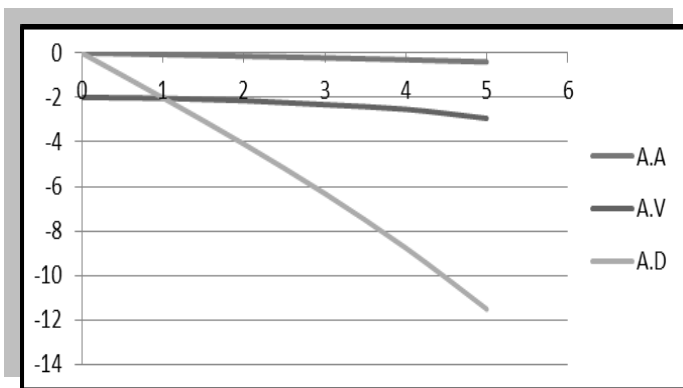Fig. 5: Angular acceleration of the pendulum as the time increases

Fig. 6: Comparison of the angular Displacement, Velocity and Acceleration of the pendulum as time increases.

V. RESULT DISCUSSION AND CONCLUSION

In this study, firstly, equation of motion of inverted pendulum was formulated using Lagrangian equation, which involves both the kinetic and potential energies of the pendulum. Then differential transform method (DTM), as a semi analytical method, was adopted to analyse the resulted second order ordinary differential equation. With the assumptions and consideration of initial conditions, the solutions obtained were very close the ones obtained when solved analytically. From Figure 1 above, it can be seen that the angular displacement increases absolutely as time increases. This is always the case especially at the initial stage. Also the angular velocity and angular acceleration in figures 2 and figure 3 respectively increase with time, but at a lower rate when compared to the rate of angular displacement. This implies that the rate of change of angular displacement and the rate of change of velocity are mild compared to change in displacement itself. A closer look at figures 2 and figure 3 shows that the angular velocity at time t = 0, is -2, while the angular acceleration is 0. In figure 4, the comparison of the dynamic effect of the pendulum over a period of time, shows that angular displacement is more pronounced than the angular velocity. The angular acceleration is less than angular velocity. The negative values of the displacement, velocity, and acceleration as time increases resulted from the fact that the pendulum under consideration is an inverted pendulum. These results are in agreement with the ones in literature. DTM is very easy to use and reduces computation time.

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