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EXISTENCE, UNIQUENESS AND STABILITY OF A MILD SOLUTION OF LIPSCHITZIAN QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

We introduce the concept of a mild solution of Lipschitzian quantum stochastic differential equations (QSDEs). Results on the existence, uniqueness and stability of a mild solution of QSDEs are established. This is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus. Here, the results on a mild solution are weaker compared with the ones in the literature.

1. Introduction

Recent literatures reveal consistent study of existence of a mild solution of differential equations ranging from classical differential equations to non-classical differential equations. For details, see [4, 5, 11, 12, 16] and the references therein.

One of the main analytical difficulties in the theory of both classical

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and non-classical stochastic differential equations (SDEs) arises when the coefficients driving the equation consist of unbounded operators, see [7]. In [6], Balasubramaniam et al. discussed the existence of mild solutions for semilinear neutral functional evolution equations with nonlocal conditions by using fractional power of operators and Kransnoselskii fixed point theorem.

In [4], existence and uniqueness of the mild solutions for stochastic differential equations for Hilbert valued stochastic processes are discussed, with the multiplicative noise term given by an integral with respect to a general compensated Poisson random measure was established.

When considering quantum stochastic differential equations (QSDEs) within the framework of Hudson and Parthasarathy [15] formulation of QSDEs not much has been done in this area. However, some properties of solution sets of quantum stochastic differential inclusions were established in [1, 2, 8].

Existence of mild solution of impulsive QSDE and SDE was also considered in [16, 18] using the fixed point theorem method. In [9], results on solution of impulsive QSDEs and the associated Kurzweil equations were established. The recurrence of such problems in the literature is the motivation for this work. Hence the results here will be an extension of the results on QSDEs in the literature.

We organize the rest of the paper as follows: in Section 2, we adopt some definitions and notations of Ekhaguere's formulations in [10] and [8, 9]. In Section 3, we introduce the QSDE with an infinitesimal generator of a family of semigroups and establish the main results on existence and uniqueness of solution. In Section 4, we establish result on stability of solution. The methods we used here are adoption of similar methods applied in [10].

All through the remaining sections, as in [8-10], we employ the locally convex topological space $\widetilde{\mathcal{A}}$ of noncommutative stochastic processes. We also adopt the definitions and notations of the following spaces $clos(\mathcal{N})$,

$clos(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma, loc}$, the integrator processes Λ_π, A^+_g, A_f, for $f, g \in L^\infty_{\gamma, loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma), loc}(\mathbb{R}_+)$.

Let $E, F, G, H \in L^2_{loc}(\tilde{A} \times I)$. The following equation is the Hudson-Parthasarathy quantum stochastic differential equation in integral form introduced in [10]:

$$X(t) = X_0 + \int_{t_0}^{t} (E(X(s), s)d \wedge_\pi (s) + F(X(s), s)dA_f(s)$$

$$+ G(X(s), s)dA^+_g(s) + H(X(s), s)ds),$$

$$X(t_0) = X_0, \quad t \in I. \tag{1.1}$$

In equation (1.1), the coefficients E, F, G and H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, annihilation processes Λ_Π, A_{f^+}, A_g and the Lebesgue measure t are defined in [10]. In the work of [10], the Hudson and Parthasarathy [15] formulation of quantum stochastic calculus was employed to establish the equivalent form of quantum stochastic differential equation (1.1) given by

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle = P(X(t), t)(\eta, \xi),$$

$$\langle \eta, X(0)\xi \rangle = \langle \eta, X_0\xi \rangle \text{ for almost all } t \in I, \tag{1.2}$$

where η, ξ lie in some dense subspaces of some Hilbert spaces which have been defined in [10]. For the explicit form of the map $P(x, t) \to P(x, t)(\eta, \xi)$ appearing in equation (1.2), see [10]. Equation (1.2) is a first order non-classical ordinary differential equation with a sesquilinear form valued map P as the right hand side. Equation (1.1) is known to have a unique weakly absolutely continuous adapted solution $\Phi : I \to \tilde{\mathcal{A}}$ for the Lipschitzian coefficients E, F, G and H.

2. Fundamental Concepts and Notations

In what follows, we employ the locally convex topological state space $\widetilde{\mathcal{A}}$ of noncommutative stochastic processes and we also adopt the definitions and notations. See [8-10] and the references therein.

Notation 2.1. In what follows, $\mathbb{D}$ is some inner product space with $\mathcal{R}$ as its completion, and γ is some fixed Hilbert space.

(i) For each $t \in \mathbb{R}_+$, we write $L^2_\gamma(\mathbb{R}_+)$ (resp. $L^2_\gamma([0, t))$; resp. $L^2_\gamma([t, \infty))$), for the Hilbert space of square integrable, γ-valued maps on $\mathbb{R}_+ \equiv [0, \infty)$ (resp. $[0, t)$; resp. $[t, \infty)$).

(ii) The noncommutative stochastic processes which we shall discuss are densely defined linear operators on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$; the inner product of this complex Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$ and its norm by $\| \cdot \|$.

Definition 2.2. For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, we define $\| \cdot \|_{\eta\xi}$ on $\mathcal{A}$ by

$$\| x \|_{\eta\xi} = | \langle \eta, x\xi \rangle |, \; x \in \mathcal{A}.$$

Then $\{\| \cdot \|_{\eta\xi}, \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is a family of seminorms on $\mathcal{A}$; we write τ_w for the locally convex Hausdorff topology on $\mathcal{A}$ determined by this family.

Notation 2.3. We denote by $\widetilde{\mathcal{A}}$ the completions of the locally convex spaces $(\mathcal{A}, \tau_w)$.

Notation 2.4. (i) We denote the space of sesquilinear forms on $\mathbb{D} \underline{\otimes} \mathbb{E}$ by $sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$.

(ii) Let $I \subseteq \mathbb{R}_+$, we denote by $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ the set of all $sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$-valued maps on I, i.e., $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E}) = \{u : I \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})\}$.

Definition 2.5. A member $z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ is:

(i) *absolutely continuous* if the map $t \to z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

(ii) of *bounded variation* if over all partitions $\{t_j\}_{j=0}^n$ of I,

$$\sup_{\mathcal{H}} \left(\sum_{j=1}^{n} | z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi) | \right) < \infty.$$

Definition 2.6. A stochastic process Φ will be called *locally absolutely p-integrable* if the map $t \to \| \Phi(t) \|_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $p \in (0, \infty)$.

Notation 2.7. For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, $L^2_{loc}(I \times \widetilde{\mathcal{A}})$ denotes the set of maps $\Phi : I \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ such that the map $t \to \Phi(X(t), t)$ lies in $L^p_{loc}(\widetilde{\mathcal{A}})$ for every $X \in L^p_{loc}(\widetilde{\mathcal{A}})$.

Definition 2.8. Let $I \subseteq \mathbb{R}_+$.

(i) A map $\Phi : I \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ will be called *Lipschitzian* if for any $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, there exists a function

$$K^{\Phi}_{\eta\xi} : I \to (0, \infty)$$

lying in $L^1_{loc}(I)$ such that

$$\| \Phi(x, t) - \Phi(y, t) \|_{\eta\xi} \leq K^{\Phi}_{\eta\xi}(t) \| x - y \|_{\eta\xi}$$

for all $x, y \in \widetilde{\mathcal{A}}$ and almost all $t \in I$.

(ii) If for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $\Phi_{\eta\xi}$ is a map from $I \times \widetilde{\mathcal{A}}$ into $sesq[\mathbb{D} \underline{\otimes} \mathbb{E}]$, then for $(x, t) \in I \times \widetilde{\mathcal{A}}$, the value of $\Phi(x, t)$ at $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ will be denoted by $\Phi(x, t)(\eta, \xi)$. Such a map will be called *Lipschitzian* if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$|\Phi(x, t)(\eta, \xi) - \Phi(y, t)(\eta, \xi)| \leq K_{\eta\xi}^{\Phi}(t)\| x - y \|_{\eta\xi}$$

for all $x, y \in \widetilde{\mathcal{A}}$ and almost all $t \in I$.

(iii) If Φ is a map from $I \times \widetilde{\mathcal{A}}$ into the $sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$, then for $(x, t) \in I \times \widetilde{\mathcal{A}}$, the value of $\Phi(x, t)$ at $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, will be called *Lipschitzian* (resp. *continuous*) if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $(x, t) \to \Phi(x, t) \cdot (\eta, \xi)$ from $I \times \widetilde{\mathcal{A}}$ to $\mathbb{C}$ is Lipschitzian (resp. continuous).

Definition 2.9. Let $I \subseteq \mathbb{R}_+$.

(i) By a stochastic process indexed by I, we mean a function on I with values in $clos(\widetilde{\mathcal{A}})$.

(ii) And for $p \in (0, \infty)$, $L_{loc}^{p}(I \times \widetilde{\mathcal{A}})$ is the set of maps $\Phi : I \times \widetilde{A} \to clos(\widetilde{\mathcal{A}})$ such that $t \to \Phi(X(t), t)$, $t \in I$ lies in $L_{loc}^{p}(\widetilde{\mathcal{A}})_{mvs}$ for every $X \in L_{loc}^{p}(\widetilde{\mathcal{A}})$.

Definition 2.10. A stochastic process $p \in Ad(\widetilde{\mathcal{A}})$ is called *simple* if there exists an increasing sequence t_n, $n = 0, 1, 2, \ldots$ with $t_0 = 0$ and $t_n \to \infty$ such that for each $n \geq 0$,

$$p(t) = p(t_n) \quad \text{and} \quad t \in [t_n, t_{n+1}).$$

For a topological space $\mathcal{N}$, let $clos(\mathcal{N})$ be the collection of all nonempty closed subsets of $\mathcal{N}$; we shall employ the Hausdorff topology on the $clos(\widetilde{\mathcal{A}})$ as defined in [10]. Also, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, we define the Hausdorff distance, $\rho(A, B)$ as in [18]. Then ρ is a metric on the $clos(\mathbb{C})$ and induces a metric topology on the space.

Definition 2.11. A map $P : \widetilde{\mathcal{A}} \times [t_0, T] \to sesq[\mathbb{D} \underline{\otimes} \mathbb{E}]$ belongs to the class $C(\widetilde{\mathcal{A}} \times [t_0, T], W)$ if for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$:

(i) $P(x, \cdot)(\eta, \xi)$ is measurable for each $x \in \widetilde{\mathcal{A}}$.

(ii) There exists a family of measurable functions $M_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ such that $\int_{t_0}^{t} M_{\eta\xi} ds < \infty$ and $| P(x, \cdot)(\eta, \xi) | \leq M_{\eta\xi}(s),\ (x, s) \in \widetilde{\mathcal{A}} \times [t_0, T]$.

(iii) There exist measurable functions $K_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ such that for each $t \in [t_0, T]$, $\int_{t_0}^{t} K_{\eta\xi} ds < \infty$, and

$$| P(x, s)(\eta, \xi) - P(y, s)(\eta, \xi) | \leq K_{\eta\xi}^{p}(s) W(\| x - y \|_{\eta\xi}).$$

For $(x, s), (y, s) \in \widetilde{\mathcal{A}} \times [t_0, T]$ and where for (i)-(iii) $W(t) = t$, and

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s) ds + \int_{t_0}^{t} K_{\eta\xi}(s) ds.$$

Notation 2.12. The class $C(\widetilde{\mathcal{A}} \times [t_0, T], W)$ denotes the class of sesquilinear form-valued maps which satisfy the Lipschitz condition and the Caratheodory conditions.

Definition 2.13. A member $z \in L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ is:

(i) *absolutely continuous* if the map $t \to z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

(ii) of *bounded variation* if over all partitions $\{t_j\}_{j=0}^{n}$ of I,

$$\sup_{\mathcal{H}} \left(\sum_{j=1}^{n} | z(t_j)(\eta, \xi) - z(t_{j-1})(\eta, \xi) | \right) < \infty,$$

(iii) of *essentially bounded variation* if z is equal almost everywhere to some member of $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ of bounded variation,

(iv) a stochastic process $x \in L^0(I, \widetilde{\mathcal{A}})$ is of bounded variation if

$$\sup_{\mathcal{H}}\left(\sum_{j=1}^{n} |\langle \eta, x(t_j)\xi\rangle - \langle \eta, x(t_{j-1})\xi\rangle|\right) < \infty$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and where supremum is taken over all partitions $\{t_j\}_{j=0}^{n}$ of *I*.

Notation 2.14. We denote by $BV(\widetilde{\mathcal{A}})$ the set of all stochastic processes of bounded variation on *I*.

Definition 2.15. For $x \in BV(\widetilde{\mathcal{A}})$, define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$Var_{[a, b]}x_{\eta\xi} = \sup_{\tau}\left(\sum_{j=1}^{n} \| x(t_j) - x(t_{j-1}) \|_{\eta\xi}\right),$$

where τ is the collection of all partitions of the interval $[a, b] \subset I$. If $[a, b] = I$, then we write $Var_I x_{\eta\xi} = Varx_{\eta\xi}$. Then $\{Varx_{\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ is a family of seminorms which generates a locally convex topology on $BV(\widetilde{\mathcal{A}})$.

Notation 2.16. (i) We denote by $\overline{BV}(\widetilde{\mathcal{A}})$ the completion of $BV(\widetilde{\mathcal{A}})$ in the said topology.

(ii) We denote by $A := BV(\widetilde{\mathcal{A}}) \cap Ad(\widetilde{\mathcal{A}})_{wac}$ the stochastic process that is weakly, absolutely continuous and of bounded variation on $[t_0, T]$. The space $A(\eta, \xi)$ equips with the norm

$$\| x \|_{PC} = \sup\{| x(t)(\eta, \xi) | : t \in I\}$$

is a Banach space.

It has been well established that the quantum stochastic differential equation (2.1) introduced by Hudson and Parthasarathy provides an essential tool in the theoretical description of physical systems, especially those arising in quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems. See [8, 9] and the references therein.

3. Existence of Solution

Let A be the infinitesimal generator of a family of semigroups $\{T(t) : t \geq 0\}$ defined in [18]. We consider the existence of a mild solution of the quantum stochastic evolution problem given by

$$\begin{aligned} dx(t) = {} & A(t)x(t) + (E(x(t), t)d \wedge_{\pi}(t) + F(x(t), t)dA_g(t) \\ & + G(x(t), t)dA_{f^+}(t) + H(x(t), t)dt), \\ & \text{almost all } t \in I = [0, T], \\ x(0) = {} & x_0. \end{aligned} \tag{3.1}$$

The equivalent form of (3.1) is then given by

$$\begin{aligned} & \frac{d}{dt}\langle \eta, x(t)\xi \rangle = A(t)x(t) + P(x(t), t)(\eta, \xi), \\ & x(t_0) = x_0, \quad t \in [t_0, T]. \end{aligned} \tag{3.2}$$

Definition 3.1. An adapted stochastic process $x \in A$ is called a *mild solution* of equation (3.1) if

$$\begin{aligned} & \langle \eta, x(t)\xi \rangle - \langle \eta, T(t)x_0\xi \rangle = \int_{t_0}^{t} T(t-s)(P(x(s), s)(\eta, \xi))ds, \\ & x(t_0) = x_0, \quad t \in [t_0, T] \end{aligned} \tag{3.3}$$

holds for every $s, t \in [t_0, T]$ identically. The map P in equation (3.2) is a sesquilinear form valued-map defined in [10]. Equation (3.1) is understood in integral form (3.3) via its solution.

Definition 3.2. Let X be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a semigroup of bounded linear operator on X if

(i) $T(0) = I$ (I is the identity operator on X).

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

The following theorem established in [10] will be useful in establishing the major result in this section.

Theorem 3.1. *Let* p, q, u, v *be simple adapted stochastic processes in* $Ad(\widetilde{\mathcal{A}})$ *and let* M *be their stochastic integral. If* $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ *with* $\eta = c \otimes e(\alpha),\ \xi = d \otimes e(\beta),\ c, d \in \mathbb{D},\ \alpha, \beta \in L^{\infty}_{\gamma, loc}(\mathbb{R}_+)$, *and* $t \geq 0$, *then*

$$\langle \eta, M(t)\xi \rangle = \int_0^t \langle \eta, \{\langle \alpha(s), \pi(s)\beta(s)\rangle_\gamma p(s) + \langle f(s), \beta(s)\rangle_\gamma q(s) + \langle \alpha(s), g(s)\rangle_\gamma u(s) + v(s)\}\xi \rangle ds. \quad (3.4)$$

Next, we establish a major result.

Theorem 3.2. *Assume that*

(i) *the coefficients* E, F, G, H *appearing in equation* (3.2) *satisfy the Lipschitz condition and belong to* $L^1_{loc}(I \times \widetilde{\mathcal{A}})$,

(ii) *there exists a constant* $N_{\eta\xi} := N$ *such that* $\| T(t) \|_{\eta\xi} \leq N_{\eta\xi}$ *for each* $t \geq 0$.

Then for any fixed point $(X_0, t_0) \in A \times I$, *there exists a unique adapted and weakly absolutely continuous mild solution* Φ *of the quantum stochastic differential equation* (3.1) *satisfying* $\Phi(t_0) = X_0$.

Proof. We first construct a τ_w-Cauchy sequence $\{\Phi_n(t)\}_{n \geq 0}$ of successive approximations of Φ in $\widetilde{\mathcal{A}}$. All through except otherwise stated $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ is arbitrary. Fix $T > t_0$, $t \in [t_0, T]$. Define $T(t)\Phi_0(t) = NX_0$, and for $n \geq 0$,

$$\Phi_{n+1}(t) = NX_0 + \int_{t_0}^t T(t-s)(E(\Phi_n(s), s)d \wedge_\pi (s) + F(\Phi_n(s), s)dA_g^+(s) + G(\Phi_n(s), s)dA_f(s) + H(\Phi_n(s), s)ds).$$

We let each $\Phi_n(t),\ n \geq 1$ define an adapted weakly absolutely continuous process in A.

By hypothesis, $E(X_0, s)$, $F(X_0, s)$, $G(X_0, s)$ and $H(X_0, s)$ belong to $\widetilde{\mathcal{A}}_s$ for $s \in [t_0, T]$ and $E(X_0, .)$, $F(X_0, .)$, $G(X_0, .)$ and $H(X_0, .)$ lie in A. Therefore, the quantum stochastic integral which defines $\Phi_1(t)$ exists for $t \in [t_0, T]$.

By equation (3.4), $\Phi_1(t)$ is weakly absolutely continuous and hence locally square integrable.

Assume now that $\Phi_n(t)$ is adapted and weakly absolutely continuous, then each $E(\Phi_n(s), s)$, $F(\Phi_n(s), s)$, $G(\Phi_n(s), s)$ and $H(\Phi_n(s), s)$ is adapted and lies in $L^2_{loc}(\widetilde{\mathcal{A}})$. Thus, $\Phi_{n+1}(t)$ is adapted and well defined. Again, by equation (3.4), $\Phi_{n+1}(t)$ is a weakly absolutely continuous process in $L^2_{loc}(\widetilde{\mathcal{A}})$. Hence we have proved our claim by induction. We consider the convergence of the successive approximations.

By equation (3.4) and the definition of the map P above, we have

$$\begin{aligned}
&\| \Phi_{n+1}(t) - \Phi_n(t) \|_{\eta\xi} \\
&= | \langle \eta, (\Phi_{n+1}(t) - \Phi_n(t))\xi \rangle | \\
&= N \left| \int_{t_0}^{t} (P(\Phi_n(s), s)(\eta, \xi) - P(\Phi_{n-1}(s), s)(\eta, \xi)) ds \right|. \qquad (1)
\end{aligned}$$

Since the coefficients E, F, G, H are Lipschitzian, the map $(x, t) \to P(X, t)(\eta, \xi)$ is also Lipschitzian and hence satisfies

$$N | P(x, t)(\eta, \xi) - P(y, t)(\eta, \xi) | \leq N K^p_{\eta\xi}(t)(\| x - y \|_{\eta\xi}),$$

$$\forall x, y \in A, \quad t \in [t_0, T].$$

Substituting the last inequality in (1), we get

$$\| \Phi_{n+1}(t) - \Phi_n(t) \|_{\eta\xi} \leq \int_{t_0}^{t} NK^p_{\eta\xi}(s)(\| \Phi_n(s) - \Phi_{n-1}(s) \|_{\eta\xi})ds. \quad (2)$$

Since the map $s \to \| \Phi_1(s) - X_0 \|_{\eta\xi}$ is continuous on $[t_0, T]$, we put

$$R_{\eta\xi} = \sup_{s\in[t_0, T]} \| \Phi_1(s) - X_0 \|_{\eta\xi}, \quad s \in [t_0, T],$$

this implies that $\| \Phi_1(s) - X_0 \|_{\eta\xi} \leq R_{\eta\xi}$. Also, let

$$M_{\eta\xi}(t) = \int_{t_0}^{t} K^p_{\eta\xi}(s)ds.$$

From (2), we have

$$\| \Phi_{n+1}(t) - \Phi_n(t) \|_{\eta\xi} \leq \frac{N(R_{\eta\xi})(M_{\eta\xi}(t))^n}{n!}, \quad n, i = 1, 2, ... \quad (3)$$

This we prove by induction as follows.

For $n = 1$, inequality (3) holds by considering (2). Assume that (3) holds for $n = k$, i.e.,

$$\| \Phi_{k+1}(t) - \Phi_k(t) \|_{\eta\xi} \leq \frac{N(R_{\eta\xi})(M_{\eta\xi}(t))^k}{k!}, \quad n = 1, 2, ... \quad (4)$$

Then, by (2),

$$\| \Phi_{k+2}(t) - \Phi_{k+1}(t) \|_{\eta\xi} \leq \int_{t_0}^{t} NK^p_{\eta\xi}(s)(\| \Phi_{k+1}(s) - \Phi_k(s) \|_{\eta\xi})ds$$

$$\leq \frac{N(R_{\eta\xi})}{k!} \int_{t_0}^{t} K^p_{\eta\xi}(s)(M_{\eta\xi}(s))^k ds \text{ by (4).}$$

By applying integration by parts on the first term, we obtain

$$\int_{t_0}^{t} K^p_{\eta\xi}(s)(M_{\eta\xi}(s))^k ds = \frac{(M_{\eta\xi}(t))^{k+1}}{k+1}. \quad (5)$$

Therefore,

$$\| \Phi_{k+2}(t) - \Phi_{k+1}(t) \|_{\eta\xi} \leq \frac{N(R_{\eta\xi})(M_{\eta\xi}(t))^{k+1}}{(k+1)!}$$

so that (3) holds for $n = k + 1$ and so holds for $n = 1, 2, 3, \ldots$.

Therefore, for any $n > k$,

$$\begin{aligned} \| \Phi_{n+1}(t) - \Phi_{k+1}(t) \|_{\eta\xi} &= N \left\| \sum_{m=k+1}^{n} (\Phi_{m+1}(t) - \Phi_m(t)) \right\|_{\eta\xi} \\ &\leq N \sum_{m=k+1}^{n} \| \Phi_{m+1}(t) - \Phi_m(t) \|_{\eta\xi} \\ &\leq N \left[\sum_{m=k+1}^{n} \frac{(R_{\eta\xi})(M_{\eta\xi}(T))^m}{m!} \right]. \end{aligned}$$

It follows that $\Phi_n(t)$ is a Cauchy sequence in $\widetilde{\mathcal{A}}$ and converges uniformly to some $\Phi(t)$. Since $\Phi_n(t)$ is adapted and weakly absolutely continuous, the same is true of $\Phi(t)$.

We now show that $\Phi(t)$ satisfies the quantum stochastic differential equation (1.1).

Since $\Phi(t_0) = X(t_0) = X_0$, we have by equation (3.4),

$$\begin{aligned} &\left\| N \int_{t_0}^{t} [E(\Phi_n(s), s) d \wedge_\pi (s) + F(\Phi_n(s), s) dA_g^+(s) \right. \\ &\quad \left. + G(\Phi_n(s), s) dA_f(s) + H(\Phi_n(s), s) ds] \right\|_{\eta\xi} \\ &- \left\| N \int_{t_0}^{t} [E(\Phi(s), s) d \wedge_\pi (s) + F(\Phi(s), s) dA_g^+(s) \right. \\ &\quad \left. + G(\Phi(s), s) dA_f(s) + H(\Phi(s), s) ds] \right\|_{\eta\xi} \\ &= N \left| \int_{t_0}^{t} (P(\Phi_n(s), s)(\eta, \xi) - P(\Phi(s), s)(\eta, \xi)) ds \right| \end{aligned}$$

$$\leq N\int_{t_0}^{t} K_{\eta\xi}^{p}(s)(\| \Phi_n(s) - \Phi(s) \|_{\eta\xi}) \to 0 \text{ as } n \to \infty,$$

since $\Phi_n(s) \to \Phi(s)$ in $\widetilde{\mathcal{A}}$ uniformly on $[t_0, T]$.

Thus,

$$\Phi(t) = \lim_{n\to\infty} \Phi_{n+1}(t)$$

$$= T(t)X_0 + \lim_{n\to\infty}\int_{t_0}^{t} T(t-s)(E(\Phi_n(s), s)d \wedge_{\pi} (s)$$

$$+ F(\Phi_n(s), s)dA_g^{+}(s) + G(\Phi_n(s), s)dA_f(s) + H(\Phi_n(s), s)ds)$$

$$= T(t)X_0 + \int_{t_0}^{t} T(t-s)(E(\Phi(s), s)d \wedge_{\pi} (s)$$

$$+ F(\Phi(s), s)dA_g^{+}(s) + G(\Phi(s), s)dA_f(s) + H(\Phi(s), s)ds).$$

That is, $\Phi(t)$, $t \in [t_0, T]$ is a solution of equation (3.3).

Uniqueness of solution

To establish the uniqueness of solution, we assume that $Y(t)$, $t \in [t_0, T]$ is another adapted weakly absolutely continuous solution with $Y(t_0) = X_0$. Then, by equation (3.4), we obtain again

$$\| \Phi(t) - Y(t) \|_{\eta\xi} = N\left| \int_{t_0}^{t} (P(\Phi(s), s)(\eta, \xi) - P(Y(s), s)(\eta, \xi))ds \right|$$

$$\leq \int_{t_0}^{t} NK_{\eta\xi}^{p}(s)(\| \Phi(s) - Y(s) \|_{\eta\xi})ds.$$

Since the integral $\int_{t_0}^{t} K_{\eta\xi}^{p}(s)$ exists on $[t_0, T]$, it is also essentially bounded on the given interval. Hence, there exists a constant $C_{\eta\xi, t}$ such that

$$\text{ess sup } K_{\eta\xi}^{p}(s) = C_{\eta\xi, t}, \quad s \in [t_0, T].$$

Thus,

$$\| \Phi(t) - Y(t) \|_{\eta\xi} \leq NC_{\eta\xi, t} \int_{t_0}^{t} (\| \Phi(s) - Y(s) \|_{\eta\xi}) ds.$$

By the Gronwall's inequality, we conclude that $\Phi(t) = Y(t),\ t \in [t_0, T]$. Hence the solution is unique.

4. Stability of Solution

The next theorem establishes that the solutions equation (3.2) is stable. Hence we let the coefficients E, F, G, H satisfy the conditions of Theorem 3.2. Let $X(t),\ Y(t),\ t \in [t_0, T]$ be solutions to equation (3.2) corresponding to the initial conditions $T(t)X(t_0) = T(t)X_0$ and $T(t)Y(t_0) = T(t)Y_0$, respectively, where $X_0, Y_0 \in A$. The solution $X(t)$ is stable under the changes in the initial condition $X(t_0) = X_0$ as follows:

Theorem 4.1. *For given* $T > t_0$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $N\| X_0 - Y_0 \|_{\eta\xi} < N\delta$, *then* $\| X(t) - Y(t) \|_{\eta\xi} < \varepsilon$ *still holds for all* $t \in [t_0, T]$ *and for each pair of* $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Proof. Let $X_n(t),\ Y_n(t),$ for $n = 0, 1, \ldots,$ be the iterates corresponding to the initial conditions X_0 and Y_0, respectively, so that $X_0(t) = X_0$ and $Y_0(t) = Y_0$ for all $t_0 \leq t \leq T$. Then we obtain the following estimate by employing the definition of P and equation (3.4) as in the proof of uniqueness of solution:

$$\begin{aligned}
&\| X_{n+1}(t) - Y_{n+1}(t) \|_{\eta\xi} \\
&\leq \| T(t)X_0 - T(t)Y_0 \|_{\eta\xi} \\
&\quad + \int_{t_0}^{t} T(t-s)(| P(X_n(s), s)(\eta, \xi) - P(Y_n(s), s)(\eta, \xi) | ds) \\
&= N\| X_0 - Y_0 \|_{\eta\xi} + N\left(C_{\eta\xi, t} \int_{t_0}^{t} \| X_n(s) - Y_n(s) \|_{\eta\xi} ds \right),
\end{aligned}$$

where $C_{\eta\xi, t}$ is the essential supremum of $K^{p}_{\eta\xi}(t)$ on $[t_0, T]$. Therefore, by iteration, we obtain for $t_0 \leq t \leq T$,

$$\begin{aligned}\| X_{n+1}(t) - Y_{n+1}(t) \|_{\eta\xi} &\leq N\| X_0 - Y_0 \|_{\eta\xi} + N\Bigg(C_{\eta\xi, t}\Bigg[\int_{t_0}^{t}\Bigg(\| X_0 - Y_0 \|_{\eta\xi} \\ &\quad + C_{\eta\xi, t_1}\int_{t_0}^{t_1} \| X_{n-1}(t_2) - Y_{n-1}(t_2) \|_{\eta\xi}\, dt_2 \Bigg) dt_1\Bigg]\Bigg) \\ &\leq N\| X_0 - Y_0 \|_{\eta\xi} + N(l_{\eta\xi}(t - t_0)\| X_0 - Y_0 \|_{\eta\xi}) \\ &\quad + N\left(l^2_{\eta\xi}\int_{t_0}^{t}\int_{t_0}^{t_1} \| X_{n-1}(t_2) - Y_{n-1}(t_2) \|_{\eta\xi}\, dt_2 dt_1\right),\end{aligned}$$

where

$$l_{\eta\xi} = \max\{C_{\eta\xi, t}, C_{\eta\xi, t_1}\}.$$

Continuing the iteration and putting

$$L_{\eta\xi} = \max\{C_{\eta\xi, t}, C_{\eta\xi, t_j}, j = 1, 2, ..., n\},$$

we obtain the estimate

$$\begin{aligned}&\| X_{n+1}(t) - Y_{n+1}(t) \|_{\eta\xi} \\ &\leq N\| X_0 - Y_0 \|_{\eta\xi} + NL_{\eta\xi}(t - t_0)\| X_0 - Y_0 \|_{\eta\xi} + \cdots \\ &\quad + NL^{n}_{\eta\xi}\frac{(t - t_0)^n}{n!}\| X_0 - Y_0 \|_{\eta\xi} \\ &\quad + NL^{n+1}_{\eta\xi}\int_{t_0}^{t}\int_{t_0}^{t_1}\cdots\int_{t_0}^{t_n} \| X_0(t_{n+1}) - Y_0(t_{n+1}) \|_{\eta\xi}\, dt_1 dt_2 \ldots dt_{n+1} \\ &\leq N\sum_{m=0}^{n+1}\frac{L^{m}_{\eta\xi}}{m!}(t - t_0)^m\| X_0 - Y_0 \|_{\eta\xi}\end{aligned}$$

$$\leq N\sum_{m=0}^{n+1}\frac{(L_{\eta\xi}T)^m}{m!}\| X_0 - Y_0 \|_{\eta\xi}$$

$$\leq N\| X_0 - Y_0 \|_{\eta\xi} e^{(L_{\eta\xi}T)}.$$

Letting $n \to \infty$, we conclude that

$$\| X(t) - Y(t) \|_{\eta\xi} \leq N\| X_0 - Y_0 \|_{\eta\xi} e^{(L_{\eta\xi}T)}$$

for all $t_0 \leq t \leq T$, and the result follows.

5. Conclusion

We have established the existence, uniqueness and stability of a mild solution of QSDE (3.1) via equation (3.2). This is possible since equivalence of these equations has been established in [10].

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