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THE EQUIVALENCE OF JUNGCK-TYPE ITERATIONS FOR GENERALIZED CONTRACTIVE-LIKE OPERATORS IN A BANACH SPACE

Abstract. We show that the convergences of Jungck, Jungck-Mann, Jungck-Ishikawa, Jungck-Noor and Jungck-multistep iteration processes are equivalent for a class of generalized contractive-like operators defined on a Banach space. Our results are generalizations and extensions of the work of Soltuz [20, 21], Zhiqun [23] and some other numerous ones in literature.

Key words: Jungck, Jungck-Mann and Jungck-Ishikawa, Jungck-Noor and Jungck- multistep iteration processes, generalized contractive-like operators, common fixed points.

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1. Introduction

Let X be a Banach space, K a nonempty convex subset of X and $T : K \to K$ a self map of K.

Definition 1. *Let $x_0 \in K$. The Picard iterative scheme $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots . \tag{1}$$

Definition 2. *Let $x_0 \in K$, the Mann iterative Scheme [10] $\{u_n\}_{n=0}^{\infty}$ is defined by*

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n, \tag{2}$$

where $\{\alpha_n\}_n^{\infty}$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Definition 3. *Let $y_0 \in K$, the Ishikawa iterative scheme [5] $\{y_n\}_{n=0}^{\infty}$ is defined by*

$$\begin{aligned} y_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Tz_n \\ z_n &= (1 - \beta_n)y_n + \beta_n Ty_n, \end{aligned} \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Definition 4. *Let $y_0 \in K$, the Noor iterative (three-step) scheme [11] $\{y_n\}_{n=0}^{\infty}$ is defined by*

$$\begin{aligned} y_{n+1} &= (1-\alpha_n)y_n + \alpha_n T z_n \\ z_n &= (1-\beta_n)y_n + \beta_n T v_n \\ v_n &= (1-\gamma_n)y_n + \gamma_n T y_n, \end{aligned} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Definition 5. *Let $y_0 \in K$. The multistep iterative scheme [21] $\{y_n\}_{n=0}^{\infty}$ is defined by*

$$\begin{aligned} y_{n+1} &= (1-\alpha_n)y_n + \alpha_n T z_n^1 \\ z_n^i &= (1-\beta_n^i)y_n + \beta_n^i T z_n^{i+1}, i = 1, 2, \ldots, k-2 \\ z_n^{k-1} &= (1-\beta_n^{k-1})y_n + \beta_n^{k-1} T y_n, \quad k \geq 2) \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}_{n=}^{\infty}$, $i = 1, 2, ...k-1$ are real sequences in [0,1) such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Remark 1. The multistep iterative scheme (5) generalizes the iterative schemes (4), (3) and (2) in that,

(i) If $k = 3$ in (5), we have the Noor iterative scheme (4).
(ii) If $k = 2$ in (5), we have the Ishikawa iterative scheme (3).
(iii) If $k = 1$ in (5), we have the Mann iterative scheme (2).

Several generalizations of the Banach fixed point theorem have been proved to date, (for example see [1], [3], [9] and [23]). One of the most commonly studied generalization hitherto is the one proved by Zamfirescu [23] in 1972, which is stated thus:

Theorem Z. *Let (X, d) be a complete metric space and $T : X \to X$ a Zamfirescu operator satisfying*

$$\begin{aligned} \|Tx - Ty\| \leq\ & h \max\{\|x-y\|, \frac{1}{2}(\|x-Tx\| + \|y-Ty\|), \\ & \frac{1}{2}(\|x-Ty\| + \|y-Tx\|)\}, \end{aligned} \tag{6}$$

where $0 \leq h < 1$. Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \tag{7}$$

$n = 0, 1, 2, \ldots$, converges to p for any $x_0 \in X$.

Several papers have been written recently on the Zamfirescu operators (6). For example (see [1], [2], [13], [18], [21-24]) . The most commonly used methods of approximating the fixed points of the Zamfirescu operators are Picard [21], Mann [10], Ishikawa [5], Noor [11] and multistep [22] iterative schemes.

Many researchers have concentrated of late on the equivalence of the various iterative schemes, that is, whether the convergence of any of the iterative method to the unique fixed point of the Zamfirescu operator is equivalent to the convergence of the other iterative schemes. For a look at some of the recent results in this area (see [21], [22] and [24]).

Jungck [6] introduced a different perspective into the generalization of Banach contraction principle, Singh et al. [20] significantly improved on the result of Jungck [6] when he proved the following result which is now called Jungck contraction principle.

Theorem Js ([20]). *Let Y be an arbitrary nonempty set and (X, d) a metric space. Let $S, T : Y \to X$ satisfy $d(Tx, Ty) \leq kd(Sx, Sy)$, $0 \leq k < 1$, for all $x, y \in Y$. $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X, then S and T have a coincidence. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that*

(a) $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$

(b) $\{Sx_n\}$ converges to Sp for some p in Y, and $Sp = Tp$, that is S and T have a coincidence at p.

Further, if $Y = X$ and S, T commute (just) at p, then S and T have a unique common fixed point.

Remark 2. If $Y = X$ and $S = id$ (identity map), then the map $d(Tx, Ty) \leq kd(Sx, Sy)$ (Jungck contraction map) is the same as the contraction map $d(Tx, Ty) \leq kd(x, y)$.

2. Preliminaries

The following definitions are useful in our work. Let X be a Banach space and Y an arbitrary set. Let $S, T : Y \to X$ be two non self mappings such that $T(Y) \subseteq S(Y)$.

Definition 6 ([6]). *For $x_0 \in Y$, the Jungck iterative scheme is a sequence $\{Sx_n\}_n^\infty$ defined by*

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots \quad . \tag{8}$$

Singh et al. [20] recently introduced the Jungck-Mann iterative process and discussed its stability for a pair of contractive maps. The iterative process is defined thus:

Definition 7 ([15]). *For $u_0 \in Y$ the Jungck-Mann iterative scheme is the sequence $\{Su_n\}_n^\infty$ defined by*

$$Su_{n+1} = (1-\alpha_n)Su_n + \alpha_n Tu_n, \quad n = 0,1,2,\dots . \tag{9}$$

where $\{\alpha_n\}_{n=0}^\infty$ is a sequence in $[0,1)$ such that $\sum_{n=0}^\infty = \infty$.

Olatinwo and Imoru [15] and Olatinwo [17] built on that work to introduce the Jungck-Ishikawa and Jungck-Noor iterative schemes and use their convergences to approximate the coincidence points of some pairs of generalized contractive-like operators with the assumption that one of each of the pairs of maps is injective. Their iterative schemes are defined as follows:

Definition 8 ([15]). *For $g_0 \in Y$, the Jungck-Ishikawa iterative scheme is the sequence $\{Sg_n\}_{n=0}^\infty$ defined by*

$$\begin{aligned} Sg_{n+1} &= (1-\alpha_n)Sg_n + \alpha_n Tz_n \\ Sz_n &= (1-\beta_n)Sg_n + \beta_n Tg_n \end{aligned} \tag{10}$$

where $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are real sequences in $[0,1)$ such that $\sum_{n=0}^\infty \alpha_n = \infty$.

Definition 9 ([17]). *Let $t_0 \in Y$, the Jungck-Noor iterative scheme is the sequence $\{St_n\}_{n=0}^\infty$ defined by*

$$\begin{aligned} St_{n+1} &= (1-\alpha_n)St_n + \alpha_n Tv_n \\ Sv_n &= (1-\beta_n)St_n + \beta_n Tw_n \\ Sw_n &= (1-\gamma_n)St_n + \gamma_n Tt_n \end{aligned} \tag{11}$$

where $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ and $\{\gamma_n\}_{n=0}^\infty$ are real sequences in $[0,1)$ such that $\sum_{n=0}^\infty \alpha_n = \infty$.

The authors [14] introduced the Jungck-multistep iterative scheme and use the convergence to approximate the common fixed points of those pairs of generalized contractive-like operators without assuming the injectivity of any of the operators but rather they proved their results for a pair of weakly compatible maps S, T.

Definition 10 ([14]). *Let $y_0 \in Y$, the Jungck-multistep iterative scheme is the sequence $\{Sy_n\}_{n=0}^\infty$ defined by*

$$\begin{aligned} Sy_{n+1} &= (1-\alpha_n)Sy_n + \alpha_n Tz_n^1 \\ Sz_n^i &= (1-\beta_n^i)Sy_n + \beta_n^i Tz_n^{i+1}, \quad i = 1,2,...k-2 \\ Sz_n^{k-1} &= (1-\beta_n^{k-1})Sy_n + \beta_n^{k-1} Ty_n, k \geq 2 \end{aligned} \tag{12}$$

where $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n^i\}_{n=0}^\infty$, $i = 1,2,\dots,k-1$ are real sequences in $[0,1)$ such that $\sum_{n=0}^\infty \alpha_n = \infty$.

Remark 3. (A) The Jungck-multistep iterative scheme (12) generalizes the iterative schemes (11), (10) and (9), that is

(i) If $k = 3$ in (12), we have the Jungck-Noor iterative scheme (11).

(ii) If $k = 2$ in (12), we have the Jungck-Ishikawa iterative scheme (10).

(iii) If $k = 2$ and $\beta_n^1 = 0$ in (12), we have the Jungck-Mann iterative scheme (9).

(B) If $X = Y$ and $S = id$ (identity map), the Jungck-multistep (12), Jungck-Noor (11), Jungck-Ishikawa (10), Jungck-Mann (9) and Jungck (8) iterative schemes reduces to multistep (5), Noor (4), Ishikawa (3), Mann (2) and Picard (1) iterative schemes respectively.

Definition 11 ([15]). *The maps $S, T : Y \to X$ with $T(Y) \subseteq S(Y)$, is called the generalized Zamfirescu operators if*

$$\begin{aligned}(13)\qquad \|Tx - Ty\| \ \le\ & h \max\{\|Sx - Sy\|, \frac{1}{2}(Sx - Tx\| + \|Sy - Ty\|), \\ & \frac{1}{2}(\|Sx - Ty\| + \|Sy - Tx\|)\},\end{aligned}$$

where $0 \le h < 1$. If $S = id$ (identity map) and $Y = X$, the generalized Zamfirescu operator (13) reduces to the Zamfirescu operator. We observe that condition (13) implies

$$(14)\qquad \|Tx - Ty\| \le \delta\|Sx - Sy\| + 2\delta\|Sx - Tx\|,$$

where $0 \le \delta < 1$ and $\delta = \max\{h, \frac{h}{2-h}\}$. For details of proof see [15].

Definition 12 ([14]). *A point $p \in X$ is called a coincident point of a pair of self maps S, T if there exist a point q (called a point of coincidence) in X such that $q = Sp = Tp$. Self maps S and T are said to be weakly compatible if they commute at their coincidence points, that is if $Sp = Tp$ for some $p \in X$, then $STp = TSp$.*

Olatinwo [16] introduced the following general contractive-like operator and define thus:

Definition 13 ([16]). *For $S, T :\to X$ with $T(Y) \subseteq S(Y)$, whenever $S(Y)$ is a complete subspace of X. There exist a real number $\delta \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and for every $x, y \in Y$, we have*

$$(15)\qquad \|Tx - Ty\| \le \delta\|Sx - Sy\| + \varphi(\|Sx - Tx\|)$$

Remark 4. If $\varphi(x) = 2\delta x$ in (15), we have (14). Hence (15) generalizes (14).

In this paper, we show that the convergence of Jungck [6], Jungck-Mann [15] and Jungck-multistep [14] iterative schemes are equivalent for the class of generalized Zamfirescu operators (14) in a Banach space. As corollaries, the convergence of the Jungck-Ishikawa [15] and Jungck-Noor [17] iterative schemes are also shown to be equivalent. Our results are generalizations and extensions of the work of Soltuz [21-22], Zhiqun [24] and other numerous results in the literature.

Lemma 1 ([21]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{e_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality $a_{n+1} \le (1-\lambda_n)a_n + e_n$, where $\lambda_n \in (0,1)$, for all $n \ge n_0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $e_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main results

Theorem 1. *Let X be a Banach space. Suppose $S, T : X \to X$ are two mappings satisfying the generalized contractive-like operator (15) such that $T(X) \subseteq S(X)$. Assume S and T are weakly compatible, then they have a unique common fixed point. Let p be the unique common point of S and T (that is $Sp = Tp = p$). If $x_0, u_0 \in X$ and define $\{Sx_n\}_{n=0}^{\infty}$ and $\{Su_n^2\}_{n=0}^{\infty}$ as sequences satisfying (8) and (9) respectively. Then the following are equivalent:*

(i) The Jungck-Mann iterative scheme (9) converges to p.

(ii) The Jungck iterative scheme (8) converges to p.

Proof. (i) implies (ii): Assume $Su_n \to p$, then using (8), (9) in (15), we have

$$\begin{aligned}(16)\qquad \|Su_{n+1} - Sx_{n+1}\| &= \|(1-\alpha_n)Su_n + \alpha_n Tu_n - Tx_n\| \\ &= \|Su_n - \alpha_n Su_n + \alpha_n Tu_n - Tu_n \\ &\qquad + Tu_n - Tx_n\| \\ &\le (1-\alpha_n)\|Su_n - Tu_n\| + \|Tu_n - Tx_n\|.\end{aligned}$$

An application of (15) with $x = u_n$, $y = x_n$ in (16) gives

$$(17)\qquad \|Tu_n - Tx_n\| \le \delta\|Su_n - Sx_n\| + \varphi(\|Su_n - Tu_n\|).$$

Substituting (17) in (16), we have

$$\begin{aligned}\|Su_{n+1} - Sx_{n+1}\| &\le (1-\alpha_n)\|Su_n - Tu_n\| + \delta\|Su_n - Sx_n\| \\ &\qquad + \varphi(\|Su_n - Tu_n\|) \\ &\le \delta\|Su_n - Sx_n\| + (1-\alpha_n)(\|Su_n - p\| \\ &\quad + \|Tu_n - Tp\|) + \varphi((\|Su_n - p\| \\ &\quad + \|Tu_n - Tp\|)).\end{aligned}$$

In view of the fact that $\|Tp - Tu_n\| \leq \delta\|Su_n - p\|$ by (15), then we have

$$\begin{aligned} (18) \quad \|Su_{n+1} - Sx_{n+1}\| &\leq \delta\|Su_n - Sx_n\| + (1-\alpha_n)(1+\delta)\|Su_n - p\| \\ &\quad + \varphi((1+\delta)\|Su_n - p\|). \end{aligned}$$

Let $a_n = \|Su_n - Sx_n\|$

$$\begin{aligned} \lambda &= 1 - \delta \\ e_n &= (1-\alpha_n)(1+\delta)\|Su_n - p\| + \varphi((1+\delta)\|Su_n - p\|). \end{aligned}$$

Hence by Lemma 1, we obtain $\lim_{n\to\infty} \|Su_n - Sx_n\| = 0$.

Therefore by

$$\|Sx_n - p\| \leq \|Sx_n - Su_n\| + \|Su_n - p\| \to 0 \quad \text{as} \quad n \to \infty,$$

we get

$$\lim_{n\to\infty} Sx_n = p. \tag{19}$$

(ii) implies (i):

$$\begin{aligned} (20) \quad \|Sx_{n+1} - Su_{n+1}\| &\leq (1-\alpha_n)\|Su_n - Tx_n\| + \alpha_n\|Tu_n - Tx_n\| \\ &\leq (1-\alpha_n)\|Su_n - Sx_n\| \\ &\quad + (1-\alpha_n)\|Sx_n - Tx_n\| + \alpha_n\|Tu_n - Tx_n\|. \end{aligned}$$

An application of (15) with $x = x_n$, $y = u_n$ in (20), gives

$$\|Tx_n - Tu_n\| \leq \delta\|Sx_n - Su_n\| + \varphi(\|Sx_n - Tx_n\|). \tag{21}$$

Substituting (21) in (20),we have

$$\begin{aligned} (22) \quad \|Sx_{n+1} - Su_{n+1}\| &\leq (1-\alpha_n)\|Sx_n - Su_n\| + (1-\alpha_n)\|Sx_n - Tx_n\| \\ &\quad + \alpha_n\delta\|Sx_n - Su_n\| + \alpha_n\varphi(\|Sx_n - Tx_n\|) \\ &= [1 - \alpha_n + \alpha_n\delta]\|Sx_n - Su_n\| \\ &\quad + (1-\alpha_n)\|Sx_n - Tx_n\| \\ &\quad + \alpha_n\varphi(\|Sx_n - Tx_n\|) \\ &\leq [1 - (1-\delta)\alpha_n]\|Sx_n - Su_n\| \\ &\quad + (1-\alpha_n)(\|Sx_n - p\| + \|Tp - Tx_n\|) \\ &\quad + \alpha_n\varphi((\|Sx_n - p\| + \|Tp - Tx_n\|)) \\ &\leq [1 - (1-\delta)\alpha_n]\|Sx_n - Su_n\| \\ &\quad + (1-\alpha_n)(1+\delta)\|Sx_n - p\| \\ &\quad + \alpha_n\varphi((1+\delta)\|Sx_n - p\|). \end{aligned}$$

Let $a_n = \|Sx_n - Su_n\|$,

$$\begin{aligned}\lambda &= 1-\delta\\ e_n &= (1-\alpha_n)(1+\delta)\|Sx_n - p\| + \alpha_n\varphi((1+\delta)\|Sx_n - p\|).\end{aligned}$$

By Lemma 1, $\sum_{n=0}^{\infty}\lambda_n = \infty$. $e_n = o(\lambda_n)$. Hence $\lim_{n\to\infty}\|Sx_n - Su_n\| = 0$. Thus

$$\|Su_n - p\| \leq \|Sx_n - Su_n\| + \|Sx_n - p\| \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore $\lim_{n\to\infty} Su_n = p$. This shows that the convergence of Jungck iterative scheme (8) is equivalent to the convergence of Jungck-Mann iterative scheme (9) for generalized contractive-like operator (15). ■

Remark 5. If $S = id$ (identity map), the proof of Theorem 2 is the same as that of Theorem 1 of Zhiqun [24].

Theorem 2. *Let X be a Banach space. Suppose $S, T : X \to X$ are two mappings satisfying the generalized contractive- like operator (15) such that $T(X) \subseteq S(X)$. Assume S and T are weakly compatible,then they have a unique common fixed point. Let p be the unique common fixed point of S and T i.e ($Sp = Tp = p$). If $u_0, y_0 \in X$ and define $\{Su_n\}_{n=0}^{\infty}$ and $\{Sy_n\}_{n=0}^{\infty}$ as sequences satisfying (9) and (12) respectively. Then the following are equivalent:*

(i) *The Jungck-Mann iterative scheme (9) converges to p.*

(ii) *The Jungck-multistep iterative scheme (12) converges to p.*

Proof. (i) implies (ii): Assume $Su_n \to p$. Using (9), (12) and (15), we have

$$\begin{aligned}\|Su_{n+1} - Sy_{n+1}\| &\leq (1-\alpha_n)\|Su_n - Sy_n\| + \alpha_n\|Tu_n - Tz_n^1\|\\ &\leq (1-\alpha_n)\|Su_n - Sy_n\| + \delta\alpha_n\|Su_n - Sz_n^1\|\\ &\quad + \alpha_n\varphi(\|Su_n - Tu_n\|).\end{aligned} \tag{23}$$

Using (15), we have

$$\begin{aligned}\|Su_n - Sz_n^1\| &\leq (1-\beta_n^1)\|Su_n - Sy_n\| + \beta_n^1\|Su_n - Tz_n^2\|\\ &\leq (1-\beta_n^1)\|Su_n - Sy_n\| + \beta_n^1\|Su_n - Tu_n\|\\ &\quad + \delta\beta_n^1\|Su_n - Sz_n^2\| + \beta_n^1\varphi(\|Su_n - Tu_n\|).\end{aligned} \tag{24}$$

Substituting (24) in (23), we have

$$\begin{aligned}\|Su_{n+1} - Sy_{n+1}\| &\leq (1-\alpha_n)\|Su_n - Sy_n\|\\ &\quad + \delta\alpha_n(1-\beta_n^1)\|Su_n - Sy_n\|\\ &\quad + \delta\alpha_n\beta_n^1\|Su_n - Tu_n\| + \delta^2\alpha_n\beta_n^1\|Su_n - Sz_n^2\|\\ &\quad + \delta\alpha_n\beta_n^1\varphi(\|Su_n - Tu_n\|) + \alpha_n\varphi(\|Su_n - Tu_n\|).\end{aligned} \tag{25}$$

$$\begin{aligned}
&= [1-(1-\delta)\alpha_n - \delta\alpha_n\beta_n^1]\|Su_n - Sy_n\| \\
&\quad + \delta^2\alpha_n\beta_n^1\|Su_n - Sz_n^2\| \\
&\quad + \delta\alpha_n\beta_n^1\|Su_n - Tu_n\| \\
&\quad + (\delta\alpha_n\beta_n^1 + \alpha_n)\varphi(\|Su_n - Tu_n\|).
\end{aligned}$$

Using (15) in (12), we have

$$\begin{aligned}
\|Su_n - Sz_n^2\| &\le (1-\beta_n^2)\|Su_n - Sy_n\| + \beta_n^2\|Su_n - Tz_n^3\| \\
&\le (1-\beta_n^2)\|Su_n - Sy_n\| + \beta_n^2\|Su_n - Tu_n\| \\
&\quad + \delta\beta_n^2\|Su_n - Sz_n^3\| + \beta_n^2\varphi(\|Su_n - Tu_n\|).
\end{aligned} \tag{26}$$

substituting (26) in (25), we have

$$\begin{aligned}
\|Su_{n+1} - Sy_{n+1}\| &\le [1-(1-\delta)\alpha_n - \delta\alpha_n\beta_n^1]\|Su_n - Sy_n\| \\
&\quad + \delta^2\alpha\beta_n^1(1-\beta_n^2)\|Su_n - Sy_n\| \\
&\quad + \delta^2\alpha_n\beta_n^1\beta_n^2\|Su_n - Tu_n\| \\
&\quad + \delta^3\alpha_n\beta_n^1\beta_n^2\|Su_n - Sz_n^3\| + \delta\alpha_n\beta_n^1\|Su_n - Tu_n\| \\
&\quad + \delta^2\alpha_n\beta_n^1\beta_n^2\varphi(\|Su_n - Tu_n\|) \\
&\quad + (\alpha_n + \delta\alpha_n\beta_n^1)\varphi(\|Su_n - Tu_n\|) \\
&= [1-(1-\delta)\alpha_n - (1-\delta)\delta\alpha_n\beta_n^1 \\
&\quad - \delta^2\alpha_n\beta_n^1\beta_n^2]\|Su_n - Sy_n\| \\
&\quad + \delta^3\alpha_n\beta_n^1\beta_n^2\|Su_n - Sz_n^3\| \\
&\quad + (\delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2)\|Su_n - Tu_n\| \\
&\quad + (\alpha_n + \delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2)\varphi(\|Su_n - Tu_n\|).
\end{aligned} \tag{27}$$

Using (15) in (12), we have

$$\begin{aligned}
\|Su_n - Sz_n^3\| &\le (1-\beta_n^3)\|Su_n - Sy_n\| + \beta_n^3\|Su_n - Tz_n^4\| \\
&\le (1-\beta_n^3)\|Su_n - Sy_n\| + \beta_n^3\|Su_n - Tu_n\| \\
&\quad + \delta\beta^3\|Su_n - Sz_n^4\| + \beta_n^3\varphi(\|Su_n - Tu_n\|).
\end{aligned} \tag{28}$$

Substituting (28) in (27), we have

$$\begin{aligned}
\|Su_{n+1} - Sy_{n+1}\| &\le [1-(1-\delta)\alpha_n - (1-\delta)\delta\alpha_n\beta_n^1 \\
&\quad - \delta^2\alpha_n\beta_n^1\beta_n^2]\|Su_n - Sy_n\| \\
&\quad + \delta^3\alpha_n\beta_n^1\beta_n^2(1-\beta_n^3)\|Su_n - Sy_n\| \\
&\quad + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\|Su_n - Tu_n\| + (\delta\alpha_n\beta_n^1 \\
&\quad + \delta^2\beta_n^2\alpha_n\beta_n^1\beta_n^2)\|Su_n - Tu_n\| \\
&\quad + \delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3\|Su_n - Sz_n^4\| \\
&\quad + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\varphi(\|Su_n - Tu_n\|) \\
&\quad + (\alpha_n + \delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2)\|Su_n - Tu_n\|
\end{aligned} \tag{29}$$

$$\begin{aligned} = {} & [1-(1-\delta)\alpha_n-(1-\delta)\delta\alpha_n\beta_n^1-(1-\delta)\delta^2\alpha_n\beta_n^1\beta_n^2 \\ & -\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3]\|Su_n-Sy_n\| \\ & +\delta^4\alpha_n\beta_n^1\beta_n^2\beta_n^3\|Su_n-Sz_n^4\| \\ & +(\delta\alpha_n\beta_n^1+\delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3)\|Su_n-Tu_n\| \\ & +(\alpha_n+\delta\alpha_n\beta_n^1+\delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3)\varphi(\|Su_n-Tu_n\|). \end{aligned}$$

Continuing the above process, we have

$$\begin{aligned} (30)\quad \|Su_{n+1}-Sy_{n+1}\| \leq {} & [1-(1-\delta)\alpha_n-(1-\delta)\delta\alpha_n\beta_n^1 \\ & -(1-\delta)\delta^2\alpha_n\beta_n^1\beta_n^2 \\ & -\ \ldots-(1-\delta)\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-3} \\ & -\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2}]\|Su_n-Sy_n\| \\ & +\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2}\|Su_n-Sz_n^{k-1}\| \\ & +(\delta\alpha_n\beta_n^1+\delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3+\ldots \\ & +\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2})\|Su_n-Tu_n\|(\alpha_n \\ & +\delta\alpha_n\beta_n^1+\delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3+\ldots \\ & +\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2})\varphi(\|Su_n-Tu_n\|). \end{aligned}$$

Using (15) in (12), we have

$$\begin{aligned} (31)\quad \|Su_n-Sz_n^{k-1}\| &\leq (1-\beta_n^{k-1})\|Su_n-Sy_n\|+\beta_n^{k-1}\|Su_n-Ty_n\| \\ &\leq (1-\beta_n^{k-1})\|Su_n-Sy_n\|+\beta_n^{k-1}\|Su_n-Tu_n\| \\ &\quad +\delta\beta_n^{k-1}\|Su_n-Sy_n\|+\beta_n^{k-1}\varphi(\|Su_n-Tu_n\|). \end{aligned}$$

Substituting (31) in (30), we have

$$\begin{aligned} (32)\quad \|Su_{n+1}-Sy_{n+1}\| \leq {} & [1-(1-\delta)\alpha_n-(1-\delta)\delta\alpha_n\beta_n^1 \\ & -(1-\delta)\delta^2\alpha_n\beta_n^1\beta_n^2 \\ & -\ \ldots-(1-\delta)\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-3} \\ & -(1-\delta)\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2} \\ & -(1-\delta)\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-1}]\|Su_n-Sy_n\| \\ & +(\delta\alpha_n\beta_n^1+\delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3+\ldots \\ & +\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2}+\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3 \\ & \ldots\beta_n^{k-1})\|Su_n-Tu_n\| \\ & +(\alpha_n+\delta\alpha_n\beta_n^1+\delta^2\alpha_n\beta_n^1\beta_n^2 \\ & +\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3+\ldots+\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-2} \\ & +\delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots\beta_n^{k-1})\varphi(\|Su_n-Tu_n\|), \end{aligned}$$

$$(33)\quad \begin{aligned}\|Su_{n+1} - Sy_{n+1}\| \leq\ & [1-(1-\delta)\alpha_n \\ & - \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-1}]\|Su_n - Sy_n\| \\ & + (\delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2 + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3 \\ & + \dots + \delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2} \\ & + \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-1})(\|Su_n - p\| \\ & + \|Tp - Tu_n\|) + (\alpha_n + \delta\alpha_n\beta_n^1 \\ & + \delta^2\alpha_n\beta_n^1\beta_n^2 + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3 + \dots \\ & + \delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2} \\ & + \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-1})\varphi(\|Su_n - p\| \\ & + \|Tp - Tu_n\|),\end{aligned}$$

$$(34)\quad \begin{aligned}\|Su_{n+1} - Sy_{n+1}\| \leq\ & [1-(1-\delta)\alpha_n]\|Su_n - Sy_n\| \\ & + (\delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2 + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3 + \dots \\ & + \delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2} \\ & + \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\cdots\beta_n^{k-1})(1+\delta)(\|Su_n - p\|) \\ & + (\alpha_n + \delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2 + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3 \\ & + \dots + \delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2} \\ & + \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-1})\varphi((1+\delta)\|Su_n - p\|).\end{aligned}$$

Therefore,

$$\|Su_{n+1} - Sy_{n+1}\| \leq [1-(1-\delta)\alpha_n]\|Su_n - Sy_n\| + e_n$$

where

$$\begin{aligned}e_n =\ & (\delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2 + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3 + \dots \\ & + \delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2} \\ & + \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3.....\beta_n^{k-1})(1+\delta)(\|Su_n - p\|) \\ & + (\alpha_n + \delta\alpha_n\beta_n^1 + \delta^2\alpha_n\beta_n^1\beta_n^2 + \delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3 \\ & + \dots + \delta^{k-2}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2} \\ & + \delta^{k-1}\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-1})\varphi((1+\delta)\|Su_n - p\|).\end{aligned}$$

Set $\lambda_n = (1-\delta)\alpha_n$, then by Lemma 1, it follows that $\lim_{n\to\infty}\|Su_n - Sy_n\| = 0$. Since $\lim_{n\to\infty} Su_n \to p$ by assumption. Then $\|Sy_n - p\| \leq \|Su_n - Sy_n\| + \|Su_n - p\| \to 0$ as $n \to \infty$. Which implies $\lim_{n\to\infty} Sy_n = p$

(ii) implies (i): Assume $Sy_n \to p$

$$(35)\quad \begin{aligned}\|Sy_{n+1} - Su_{n+1}\| \leq\ & (1-\alpha_n)\|Sy_n - Su_n\| + \alpha_n\|Tz_n^1 - Tu_n\| \\ \leq\ & (1-\alpha_n)\|Sy_n - Su_n\| + \alpha_n\delta\|Sz_n^1 - Su_n\| \\ & + \alpha_n\varphi(\|Sz_n^1 - Tz_n^1\|),\end{aligned}$$

$$\begin{aligned}(36)\qquad \|Sz_n^1 - Su_n\| &\le (1-\beta_n^1)\|Sy_n - Su_n\| + \beta_n^1\|Tz_n^2 - Su_n\| \\ &\le (1-\beta_n^1)\|Sy_n - Su_n\| + \beta_n^1\|Tz_n^2 - Sy_n\| \\ &\quad + \beta_n^1\|Sy_n - Su_n\| \\ &\le \|Sy_n - Su_n\| + \beta_n^1\|Tz_n^2 - Sy_n\|.\end{aligned}$$

Substituting (36) in (35), we have

$$\begin{aligned}(37)\quad \|Sy_{n+1} - Su_{n+1}\| &\le (1-\alpha_n)\|Sy_n - Su_n\| + \alpha_n\delta\|Sy_n - Su_n\| \\ &\quad + \alpha_n\beta_n^1\delta\|Tz_n^2 - Sy_n\| + \alpha_n\varphi(\|Sz_n^1 - Tz_n^1\|) \\ &= [1-(1-\delta)\alpha_n]\|Sy_n - Su_n\| \\ &\quad + \delta\alpha_n\beta_n^1\|Tz_n^2 - Sy_n\| + \alpha_n\varphi(\|Sz_n^1 - Tz_n^1\|) \\ &\le [1-(1-\delta)\alpha_n]\|Sy_n - Su_n\| \\ &\quad + \delta\alpha_n\beta_n^1(\|Tz_n^2 - Tp\| + \|p - Sy_n\|) \\ &\quad + \alpha_n\varphi(\|Sz_n^1 - Tz_n^1\|).\end{aligned}$$

An application of (15) with $x = p$, $y = z_n^2$ gives

$$(38)\qquad \|Tz_n^2 - Tp\| \le \delta\|Sp - Sy_n\| + \varphi(\|Sp - Tp\|) = \delta\|Sp - Sy_n\|.$$

Substituting (38) in (37), we have

$$\begin{aligned}(39)\qquad \|Sy_{n+1} - Su_{n+1}\| &\le [1-(1-\delta)\alpha_n]\|Sy_n - Su_n\| \\ &\quad + \delta\alpha_n\beta_n^1(\delta+1)\|Sy_n - p\| \\ &\quad + \alpha_n\varphi(\|Sz_n^1 - Tz_n^1\|).\end{aligned}$$

We note that $\beta_n^i \in [0,1)$, for $n = 1, 2, \ldots$ and $1 \le i \le k-1$. Therefore by (12) and (15), we have

$$\begin{aligned}(40)\quad \|Sz_n^1 - Tz_n^1\| &\le \|Sz_n^1 - p\| + \|p - Tz_n^1\| \\ &\le (\delta+1)\|Sz_n^1 - p\| \\ &\le (\delta+1)[(1-\beta_n^1)\|Sy_n - p\| + \beta_n^1\|Tz_n^2 - p\|] \\ &\le (\delta+1)[(1-\beta_n^1)\|Sy_n - p\| + \delta\beta_n^1\|Sz_n^2 - p\|] \\ &\le (\delta+1)[\|Sy_n - p\| + \delta\|Sz_n^2 - p\|] \\ &\le (\delta+1)[\|Sy_n - p\| + \|Sz_n^2 - p\|] \\ &\le (\delta+1)[\|Sy_n - p\| + (1-\beta_n^2)\|Sy_n - p\| \\ &\quad + \beta_n^2\|Tz_n^3 - p\|] \\ &\le (\delta+1)[\|Sy_n - p\| + \|Sy_n - p\| + \|Tz_n^3 - p\|] \\ &\le (\delta+1)[2\|Sy_n - p\| + \delta\|Sz_n^3 - p\|] \\ &\le (\delta+1)[2\|Sy_n - p\| + \|Sz_n^3 - p\|]\ldots \\ &\le (\delta+1)[(k-2)\|Sy_n - p\| + \|Sz_n^{k-1} - p\|]\end{aligned}$$

$$
\begin{aligned}
&\le (\delta+1)[(k-2)\|Sy_n-p\|+(1-\beta_n^{k-1})\|Sy_n-p\| \\
&\quad +\beta_n^{k-1}\|Ty_n-p\|] \\
&\le (\delta+1)[(k-2)\|Sy_n-p\|+\|Sy_n-p\|+\|Ty_n-p\|] \\
&\le (\delta+1)[(k-1)\|Sy_n-p\|+\delta\|Sy_n-p\|] \\
&\le (\delta+1)[(k-1)\|Sy_n-p\|+\|Sy_n-p\|] \\
&\le (\delta+1)k\|Sy_n-p\|.
\end{aligned}
$$

Substituting (40) in (39), we have

$$
\begin{aligned}
\|Sy_{n+1}-Su_{n+1}\| \le\ & [1-(1-\delta)\alpha_n]\|Sy_n-Su_n\| \\
& +\delta\alpha_n\beta_n^1(\delta+1)\|Sy_n-p\|+\alpha_n\varphi(k(\delta+1)\|Sy_n-p\|).
\end{aligned}
$$

Let $\lambda_n=(1-\delta)\alpha_n$ and $e_n=\delta\alpha_n\beta_n^1(\delta+1)\|Sy_n-p\|+\alpha_n\varphi(k(\delta+1)\|Sy_n-p\|$. Then,

$$\|Sy_{n+1}-Su_{n+1}\|\le[1-\lambda_n]\|Sy_n-Su_n\|+e_n.$$

Since by assumption $\lim_{n\to\infty}Sy_n=p$, then by Lemma 1,

$$\lim_{n\to}\|Sy_n-Su_n\|=0.$$

Therefore, $\|Su_n-p\|\le\|Sy_n-Su_n\|+\|Sy_n-p\|\to 0$, as $n\to\infty$ which implies $\lim_{n\to\infty}Su_n=p$. This shows that the convergence of Jungck-Mann iterative scheme (9) is equivalent to the convergence of Jungck-multistep iterative scheme (12) when applied to the generalized contractive-like operator (15). ■

Theorems 2 and 3 lead to the following Corollaries.

Corollary 1. *Let X be a Banach space. Suppose $S,T:X\to X$ are two mappings satisfying (15) such that $T(X)\subseteq S(X)$. Assume S and T are weakly compatible, then they have a unique common fixed point. Let p be the unique common fixed point of S and T (i.e $Sp=Tp=p$). If $u_0,g_0,t_0\in X$ and define $\{Su_n\}_{n=0}^{\infty}$, $\{Sg_n\}_{n=0}^{\infty}$ and $\{St_n\}_{n=0}^{\infty}$ as sequences satisfying (9), (10) and (11) respectively. Then the following are equivalent:*

a) (i) The Jungck-Mann iterative scheme (9) converges to p.
(ii) The Jungck-Ishikawa iterative scheme (10) converges to p.

b) (i) The Jungck-Mann iterative scheme (9) converges to p.
(ii) The Jungck-Noor iterative scheme (11) converges to p.

Proof. The proof of Corollary 1 is similar to that of Theorem 2. ■

Corollary 2. *Let X be a Banach space. Suppose $S,T:X\to X$ are two mappings satisfying (15) such that $T(X)\subseteq S(X)$. Assume S and T*

are weakly compatible, then they have a unique common fixed point. Let p *be the unique common fixed point of* S *and* T *(i.e* $Sp = Tp = p$*). If* $x_0, u_0, g_0, t_0, y_0 \in X$ *and define* $\{Sx_n\}_{n=0}^{\infty}$, $\{Su_n\}_{n=0}^{\infty}$, $\{Sg_n\}_{n=0}^{\infty}$, $\{St_n\}_{n=0}^{\infty}$ *and* $\{Sy_n\}_{n=0}^{\infty}$ *as sequences satisfying (8), (9), (10), (11), (12) and (13) respectively, then the following are equivalent:*

(*i*) *The Jungck-Mann iterative scheme (9) converges to* p*.*
(*ii*) *The Jungck-Ishikawa iterative scheme (10) converges to* p*.*
(*iii*) *The Jungck-Noor iterative scheme (11) converges to* p*.*
(*iv*) *The Jungck-multistep iterative (12) converges to* p*.*
(*v*) *The Jungck iterative scheme (8) converges to* p*.*

Remark 6. Theorem 2 is a generalization and extension of Theorem 1 of Soltuz [20] and Theorem 1.1 of Zhiqun [23]. Theorem 3 also generalizes and extends Theorem 2 of Soltuz [21] and some other numerous results in literature.

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