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Strong convergence and stability of Kirk-multistep-type iterative schemes for contractive-type operators

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Abstract

In this paper, we introduce Kirk-multistep and Kirk-multistep-SP iterative schemes and prove their strong convergences and stabilities for contractive-type operators in normed linear spaces. By taking numerical examples, we compare the convergence speed of our schemes (Kirk-multistep-SP iterative schemes) with the others (Kirk-SP, Kirk-Noor, Kirk-Ishikawa, Kirk-Mann and Kirk iterative schemes) for this class of operators. Our results generalize and extend most convergence and stability results in the literature.

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1 Introduction and preliminary definitions

The interest in approximating fixed points of various contractive-type operators is increasing. This is because of the close relationship that exists between the problem of solving a nonlinear equation and that of approximating a fixed point of a corresponding contractive-type operator. In 1989, Glowinski and Le-Tallec [1] used a three-step iterative process to solve elastoviscoplasticity, liquid crystal, and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge *et al.* [2] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [1] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also leads to highly parallelized algorithms under certain conditions. Hence, we can say that a multistep iterative scheme plays an important role in solving various problems in pure and applied sciences. However, there are several iterative schemes in the literature for which the fixed points of operators have been approximated over the years by various authors. For example, in a complete metric X with $x_0 \in X$, the Picard iterative sequence $\{x_n\}_{n=1}^{\infty}$ is defined by

$$x_{n+1} = Tx_n, \quad n \geq 0, \tag{1.1}$$

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has been employed to approximate the fixed points of the mapping satisfying the inequality $d(Tx, Ty) \leq ad(x, y)$, for all $x, y \in X$ and $0 \leq a < 1$.

We shall also need the following iterative schemes which appear in [3–14], and [15] to establish our results.

Let E be a normed linear space and $T : E \to E$ a self-map of E. For $x_0 \in E$, the sequence $\{x_n\}_{n=1}^{\infty}$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of positive numbers in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$; it is called the Mann iterative scheme [10].

If $\alpha_n = 1$ in (1.2), we have the Picard iterative scheme (1.1).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 0, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$; it is called an Ishikawa iterative scheme [8].

Observe that if $\beta_n = 0$ for each n, then the Ishikawa iterative scheme (1.3) reduces to the Mann iterative scheme (1.2).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \quad n \geq 0, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of positive numbers in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$; it is called the Noor iterative (or three-step) scheme [11].

Also observe that if $\gamma_n = 0$ for each n, then the Noor iteration process (1.4) reduces to the Ishikawa iterative scheme (1.3).

For $x_0 \in E$, $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n^1, \\ y_n^i &= \left(1 - \beta_n^i\right)x_n + \beta_n^i Ty_n^{i+1}, \quad i = 1, 2, \ldots, k-2, \\ y_n^{k-1} &= \left(1 - \beta_n^{k-1}\right)x_n + \beta_n^{k-1} Tx_n, \quad k \geq 2, n \geq 0, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^i\}$, $i = 1, 2, \ldots, k-1$ are sequences of positive numbers in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$; it is called a multistep iterative scheme [15].

Observe that the multistep iterative scheme (1.5) is a generalization of the Noor, the Ishikawa, and the Mann iterative schemes. In fact, if $k = 3$ in (1.5), we have the Noor iterative scheme (1.4), if $k = 2$ in (1.5), we have the Ishikawa iteration (1.3), and if $k = 2$ and $\beta_n^1 = 0$ in (1.5), we have the Mann iterative scheme (1.2).

In [12], Phuengrattana and Suantai defined the SP iterative scheme thus: For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^\infty$ is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n, \quad n \geq 0, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$, $\{\gamma_n\}_{n=0}^\infty$ are sequences of positive numbers in $[0,1]$ such that $\sum_{n=0}^\infty \alpha_n = \infty$.

The following Kirk and Kirk-type iterative schemes are of interest in this work. In [9], Kirk introduced the Kirk iterative scheme as follows: For $x_0 \in E$, the sequence $\{x_n\}_{n=1}^\infty$ is defined by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n, \quad n \geq 0, \sum_{i=0}^{k} \alpha_i = 1. \tag{1.7}$$

In [16], Olatinwo introduced the following Kirk-Mann and Kirk-Ishikawa iterative schemes: For $x_0 \in E$, the sequence $\{x_n\}_{n=1}^\infty$ is defined by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,i} T^i x_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, n \geq 0, \tag{1.8}$$

where $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0,1]$, and k is a fixed integer; it is called the Kirk-Mann iterative scheme. We have

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{k} \alpha_i T^i x_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, \\ y_n &= \sum_{i=0}^{k} \beta_{n,j} T^j x_n, \quad \sum_{j=0}^{s} \beta_{n,j} = 1, n \geq 0, \end{aligned} \tag{1.9}$$

where $k \geq s$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\alpha_{n,i}\beta_{n,j} \in [0,1]$, and k, s are fixed integers; it is called the Kirk-Ishikawa iterative scheme.

Recently, Chugh and Kumar [4] introduced the Kirk-Noor iterative scheme as follows: For $x_0 \in E$, the sequence $\{x_n\}_{n=1}^\infty$ is defined by

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{k} \alpha_i T^i y_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, \\ y_n &= \beta_{n,0} x_n + \sum_{j=1}^{s} \beta_{n,j} T^j z_n, \quad \sum_{j=0}^{s} \beta_{n,j} = 1, \\ z_n &= \sum_{l=0}^{t} \gamma_{n,l} T^l x_n, \quad \sum_{l=0}^{t} \gamma_{n,l} = 1, n \geq 0, \end{aligned} \tag{1.10}$$

where $k \geq s \geq t$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,l} \geq 0$, $\gamma_{n,0} \neq 0$, $\alpha_{n,i}, \beta_{n,j}, \gamma_{n,l} \in [0,1]$, and k, s, and t are fixed integers.

The Kirk-SP scheme [5] is defined as follows: let E be a normed linear space, $T : E \to E$ a self-map of E and $x_0 \in E$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} y_n + \sum_{i=1}^{k} \alpha_i T^i y_n, \quad \sum_{i=0}^{k} \alpha_{n,i} = 1, \\ y_n &= \beta_{n,0} z_n + \sum_{j=1}^{s} \beta_{n,j} T^j z_n, \quad \sum_{j=0}^{s} \beta_{n,j} = 1, \\ z_n &= \sum_{l=0}^{t} \gamma_{n,l} T^l x_n, \quad \sum_{l=0}^{t} \gamma_{n,l} = 1, n \geq 0, \end{aligned} \tag{1.11}$$

where $k \geq s \geq t$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,l} \geq 0$, $\gamma_{n,0} \neq 0$, $\alpha_{n,i}, \beta_{n,j}, \gamma_{n,l} \in [0,1]$, and k, s, and t are fixed integers.

In [17], Berinde showed that the Picard iteration is faster than the Mann iteration for quasi-contractive operators satisfying (1.14). In [13], Qing and Rhoades by taking an example showed that the Ishikawa iteration is faster than the Mann iteration for a certain quasi-contractive operator. Recently, Hussain *et al.* [6], provided an example of a quasi-contractive operator for which the iterative scheme due to Agarwal *et al.* is faster than Mann and Ishikawa iterative schemes.

In 2012, the authors [5] provided an example to show the rate of convergence of the Kirk-type iterative schemes. The decreasing order of Kirk-type is as follows: Kirk-SP, Kirk-CR, Kirk-Noor, Kirk-Ishikawa, and Kirk-Mann iterative scheme. However, after interchanging the parameters the decreasing order of Kirk-type iterative schemes is as follows: Kirk-CR, Kirk-SP, Kirk-Noor, Kirk-Ishikawa, and Kirk-Mann.

Several generalizations of the Banach fixed point theorem have been proved to date (for example, see [18–20] and [21]). One of the most commonly studied generalizations hitherto is the one proved by Zamfirescu [21] in 1972, which can be stated as follows.

Theorem 1.1 *Let X be a complete metric space and $T : X \to X$ a Zamfirescu operator satisfying*

$$\begin{aligned} d(Tx, Ty) \leq h \max\biggl\{ & d(x,y), \frac{1}{2}\bigl[d(x, Tx) + d(y, Ty)\bigr], \\ & \frac{1}{2}\bigl[d(x, Ty) + d(y, Tx)\bigr]\biggr\}, \end{aligned} \tag{1.12}$$

where $0 \leq h < 1$. Then T has a unique fixed point and the Picard iteration (1.1) *converges to p for any $x_0 \in X$.*

Observe that in a Banach space setting, condition (1.12) *implies*

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|, \tag{1.13}$$

where $0 \leq \delta < 1$ and $\delta = \max\{h, \frac{h}{2-h}\}$; for details of the proof see [22].

Several papers have been written on the Zamfirescu operators (1.9); for example see [14, 17, 23], and [21]. The most commonly used methods of approximating the fixed points of the Zamfirescu operators are the Picard, Mann [10], Ishikawa [8], Noor [11], and multistep [15] iterative schemes. Berinde [23] proved the convergence of the Ishikawa and Mann iterative schemes in an arbitrary Banach space. Rafiq [14], proved the convergence of the Noor iterative scheme using the Zamfirescu operators defined by (1.13).

The first important result on T-stable mappings was established by Ostrowski [24] for the Picard iterative scheme. Berinde [22] also gave the following remarkable explanation on the stability of iteration procedures.

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an iteration procedure involving the operator T,

$$x_{n+1} = f(T, x_n), \tag{1.14}$$

$n = 0, 1, 2, \ldots,$ where $x_0 \in X$ is the initial approximation and f is some function. For example, the Picard iteration (1.1) is obtained from (1.14) for $f(T, x_n) = Tx_n$, while the Mann iteration (1.2) is obtained for $f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n$, with $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ and X a normed linear space. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. When calculating $\{x_n\}_{n=0}^{\infty}$, then we cover the following steps:

1. We choose the initial approximation $x_0 \in X$.
2. Then we compute $x_1 = f(T, x_0)$, but due to various errors (rounding errors, numerical approximations of functions, derivatives or integrals), we do not get the exact value of x_1 but a different one, z_1, which is very close to x_1.
3. Consequently, when computing $x_2 = f(T, x_1)$ we shall have actually $x_2 = f(T, z_1)$ and instead of the theoretical value x_2, we shall obtain a closed value and so on. In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iterative method, we get an approximant sequence $\{z_n\}_{n=0}^{\infty}$. We say that the iteration method is stable if and only if for z_n close enough to x_n, $\{z_n\}_{n=0}^{\infty}$ still converges to the fixed point p of T. Following this idea, Harder and Hicks [25] introduced the following concept of stability.

Several authors obtained interesting stability results in literature (see, *e.g.* Akewe [26], Berinde [27], Rhoades [28–30]. Akewe and Olaoluwa [31] established some convergence results for a generalized contractive-like operators. Their results improves the results of Rafiq [32]. Olaleru and Akewe [33] extended the well known Gregus fixed point theorem.

Definition 1.2 Let (X, d) be a metric space and $T : X \to X$ a self-map, $x_0 \in X$, and the iteration procedure defined by (1.14) such that the generated sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{z_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X, and set $\epsilon_n = d(z_{n+1}, f(T, z_n))$, for $n \geq 0$. We say the iteration procedure (1.14) is T-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} z_n = p$.

In 2003, Imoru and Olatinwo [7] proved some stability results by employing the following general contractive definition:

for each $x, y \in X$, there exist $a \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + \varphi(\|x - Tx\|). \tag{1.15}$$

Several other stability results exist in the literature (for details see [16, 20, 22, 24, 25, 34, 35], and [36]).

We shall need the following lemma, which appears in [23], to prove our results.

Lemma 1.3 [23] *Let δ be a real number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, $n = 0, 1, 2, \dots$, we have $\lim_{n\to\infty} u_n = 0$.*

Lemma 1.4 [16] *Let X, $\|\cdot\|$ be a normed linear space and $T : X \to X$ be a self-map of X satisfying* (1.13). *Let $\varphi : R^+ \to R^+$ be a subadditive, monotone increasing function such that $\varphi(0) = 0$, $\varphi(Lu) = L\varphi(u)$, $L \geq 0$, $u \in R^+$. Then, for all $i \in N$, $L \geq 0$, and for all $x, y \in X$,*

$$\left\| T^i x - T^i y \right\| \leq a^i \|x - y\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\left(\|x - Tx\|\right). \tag{1.16}$$

In line with this research work of Hussain *et al.* [5], we shall introduce Kirk-multistep and Kirk-multistep-SP iterative schemes in the main result as general formulas for obtaining other Kirk-type schemes and prove that our scheme converges to the fixed point of contractive-type operators (1.16). It has already been shown [5] that Kirk-SP and Kirk-CR is better than the others. We shall only show through numerical examples that our scheme (Kirk-multistep-SP iterative schemes) converges faster than the Kirk-SP scheme for different functions.

2 Main result I

Let E be a normed linear space, $T : E \to E$ a self-map of E and $x_0 \in E$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned}
x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{k_1} \alpha_{n,i} T^i y_n^1, \quad \sum_{i=0}^{k_1} \alpha_{n,i} = 1, \\
y_n^j &= \beta_{n,0}^j x_n + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j T^i y_n^{j+1}, \quad \sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1, j = 1, 2, \dots, q-2, \\
y_n^{q-1} &= \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} T^i x_n, \quad \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1, q \geq 2, n \geq 0,
\end{aligned} \tag{2.1}$$

where $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_q$, for each j, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j}^j \geq 0$, $\beta_{n,0}^j \neq 0$, for each j, $\alpha_{n,i}, \beta_{n,i}^j \in [0,1]$ for each j and k_1, k_j are fixed integers (for each j). Equation (2.1) is called the Kirk-multistep iterative scheme.

Also, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned}
x_{n+1} &= (1 - \alpha_n) y_n^1 + \alpha_n T y_n^1, \\
y_n^j &= \left(1 - \beta_n^j\right) y_n^{j+1} + \beta_n^j T y_n^{j+1}, \quad j = 1, 2, \dots, q-2, \\
y_n^{q-1} &= \left(1 - \beta_n^{q-1}\right) x_n + \beta_n^{q-1} T x_n, \quad q \geq 2, n \geq 0,
\end{aligned} \tag{2.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^j\}$, $j = 1, 2, \dots, k-1$ are sequences of positive numbers in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$; it is called the multistep-SP iterative scheme.

Finally, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$
\begin{aligned}
x_{n+1} &= \alpha_{n,0} y_n^1 + \sum_{i=1}^{k_1} \alpha_{n,i} T^i y_n^1, \quad \sum_{i=0}^{k_1} \alpha_{n,i} = 1, \\
y_n^j &= \beta_{n,0}^j y_n^{j+1} + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j T^i y_n^{j+1}, \quad \sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1, j = 1, 2, \ldots, q-2, \\
y_n^{q-1} &= \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} T^i x_n, \quad \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1, q \geq 2, n \geq 0,
\end{aligned}
\tag{2.3}
$$

where $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_q$, for each j, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j}^j \geq 0$, $\beta_{n,0}^j \neq 0$, for each j, $\alpha_{n,i}, \beta_{n,i}^j \in [0,1]$ for each j and k_1, k_j are fixed integers (for each j). Equation (2.3) is called the Kirk-multistep-SP iterative scheme.

Remark 2.1 (i) The Kirk-multistep (2.1) is a generalization of the Kirk-Noor, Kirk-Ishikawa, Kirk-Mann, and Kirk iterative schemes; in fact if $q = 3$ in (2.1), we have the Kirk-Noor iterative scheme. If $q = 2$ in (2.1), we obtain the Kirk-Ishikawa iterative scheme and if $q = 2$ and $k_2 = 0$,we obtain the Kirk-Mann iterative scheme.

(ii) Putting $t = s = k = 1$ in (1.10), we obtain the Noor iterative scheme (1.4) with $\sum_{i=0}^{1} \alpha_{n,i} = \sum_{j=0}^{1} \beta_{n,j} = \sum_{l=0}^{1} \gamma_{n,l} = 1$, $\alpha_{n,1} = \alpha_n$, $\beta_{n,1} = \beta_n$, $\gamma_{n,1} = \gamma_n$.

(iii) Putting $t = 0$, $s = k = 1$ in (1.10), we obtain the Ishikawa iterative scheme (1.3) with $\sum_{i=0}^{1} \alpha_{n,i} = \sum_{j=0}^{1} \beta_{n,j} = 1$, $\alpha_{n,1} = \alpha_n$, $\beta_{n,1} = \beta_n$.

(iv) Putting $t = s = 0$, $k = 1$ in (1.10), we obtain the Mann iterative scheme (1.2) with $\sum_{i=0}^{1} \alpha_{n,i} = 1$, $\alpha_{n,1} = \alpha_n$.

(v) Putting $t = s = 0$, $k = 1$ and $\alpha_{n,i} = \alpha_i$ in (1.10), we obtain the Kirk iterative scheme (1.7).

(vi) Observe also that Kirk-multistep-SP iterative scheme (2.3) is a generalization of the Kirk-SP iterative schemes; if $q = 3$ in (2.3), we have the Kirk-SP iterative scheme (1.11).

(vii) Putting $q = 3$ in (2.2) we obtain the SP iterative scheme (1.6).

Some strong convergence results in normed linear spaces

Theorem 2.2 *Let $(E, \|\cdot\|)$ be a normed linear space, $T : E \to E$ be a self-map of E satisfying the contractive condition*

$$
\left\| T^i x - T^i y \right\| \leq a^i \|x - y\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\big(\|x - Tx\|\big), \tag{2.4}
$$

for each $x, y \in E$, $0 \leq a^i < 1$, and let $\varphi : R^+ \to R^+$ be a subadditive monotone increasing function with $\varphi(0) = 0$ and $\varphi(Lu) = L\varphi(u)$ $L \geq 0$, $u \in R^+$. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Kirk-multistep iterative scheme defined by (2.1). *Then*

(i) *T defined by* (2.4) *has a unique fixed point p;*

(ii) *the Kirk-multistep iterative scheme converges strongly to p of T.*

Proof (i) We shall first establish that the mapping T satisfying the contractive condition (1.16) or (2.4) has a unique fixed point.

Suppose there exist $p_1, p_2 \in F_T$, and that $p_1 \neq p_2$, with $\|p_1 - p_2\| > 0$, then

$$0 < \|p_1 - p_2\| = \left\| T^i p_1 - T^i p_2 \right\| \leq \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi^j \left(\|p_1 - Tp_1\|\right) + a^i \|p_1 - p_2\|$$

$$= \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi^j(0) + a^i \|p_1 - p_2\|.$$

Thus,

$$\left(1 - a^i\right) \|p_1 - p_2\| \leq 0.$$

Since $a \in [0,1)$, we have $1 - a^i > 0$ and $\|p_1 - p_2\| \leq 0$. Since the norm is nonnegative we have $\|p_1 - p_2\| = 0$. That is, $p_1 = p_2 = p$ (say). Thus, T has a unique fixed point p.

Next we shall establish that $\lim_{n\to\infty} x_n = p$. That is, we show that the Kirk-multistep iterative scheme converges strongly to p of T.

In view of (2.1) and (2.4), we have

$$\|x_{n+1} - p\| \leq \alpha_{n,0} \|x_n - p\| + \sum_{i=1}^{k_1} \alpha_{n,i} \left\| T^i y_n^1 - Tp \right\|. \tag{2.5}$$

Using (1.16), with $y = y_n^1$, gives

$$\left\| Tp - T^i y_n^1 \right\| \leq a^i \left\| y_n^1 - p \right\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\left(\|p - Tp\|\right). \tag{2.6}$$

Substituting (2.6) in (2.5), we have

$$\|x_{n+1} - p\| \leq \alpha_{n,0} \|x_n - p\| + \left(\sum_{i=1}^{k_1} \alpha_{n,i} a^i \right) \left\| y_n^1 - p \right\|. \tag{2.7}$$

We note that $\beta_{n,i}^j \in [0,1]$ for each j, and k_1, k_j are fixed integers (for each j), for $n = 1, 2, \ldots$ and $1 \leq j \leq q-1$. We have

$$\begin{aligned}
\left\| y_n^1 - p \right\| &\leq \beta_{n,0}^1 \|x_n - p\| + \sum_{i=1}^{k_2} \beta_{n,i}^1 \left\| T^i y_n^2 - Tp \right\| \\
&\leq \beta_{n,0}^1 \|x_n - p\| + \sum_{i=1}^{k_2} \beta_{n,i}^1 a^i \left\| y_n^2 - p \right\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\left(\|p - Tp\|\right) \\
&= \beta_{n,0}^1 \|x_n - p\| + \left(\sum_{i=1}^{k_2} \beta_{n,i}^1 a^i \right) \left\| y_n^2 - p \right\| \\
&\leq \beta_{n,0}^1 \|x_n - p\| + \left(\sum_{i=1}^{k_2} \beta_{n,i}^1 a^i \right) \left[\beta_{n,0}^2 \|x_n - p\| + \left(\sum_{i=1}^{k_3} \beta_{n,i}^2 a^i \right) \left\| y_n^3 - p \right\| \right] \\
&= \beta_{n,0}^1 \|x_n - p\| + \left(\sum_{i=1}^{k_2} \beta_{n,i}^1 a^i \right) \beta_{n,0}^2 \|x_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\|y_n^3-p\| \\
&\le \beta_{n,0}^1\|x_n-p\|+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\beta_{n,0}^2\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\beta_{n,0}^3\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3 a^i\right)\|y_n^4-p\| \\
&\le \beta_{n,0}^1\|x_n-p\|+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\beta_{n,0}^2\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\beta_{n,0}^3\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3 a^i\right)\beta_{n,0}^4\|x_n-p\|+\cdots \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3 a^i\right)\times\cdots \\
&\times\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2} a^i\right)\left(\sum_{i=1}^{k_q}\beta_{n,i}^{q-1} a^i\right)\beta_{n,0}^q\|x_n-p\|; \qquad (2.8)
\end{aligned}
$$

(2.8) holds, since $Tp = p$ and $\varphi(0) = 0$.

Substituting (2.8) in (2.7), we have

$$
\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_{n,0}\|x_n-p\|+\left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\Bigg[\beta_{n,0}^1\|x_n-p\|+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\beta_{n,0}^2\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\beta_{n,0}^3\|x_n-p\|+\cdots \\
&+\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3 a^i\right)\beta_{n,0}^4\times\cdots \\
&\times\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2} a^i\right)\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q} a^i\right)\beta_{n,0}^q\|x_n-p\|\Bigg] \\
&= \alpha_{n,0}\|x_n-p\|+\left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\beta_{n,0}^1\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\beta_{n,0}^2\|x_n-p\| \\
&+\left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\beta_{n,0}^3\|x_n-p\|+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2 a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3 a^i\right)\beta_{n,0}^4\times\cdots \\
&\times\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2}a^i\right)\left(\sum_{i=1}^{k_q}\beta_{n,i}^{q-1}a^i\right)\beta_{n,0}^q\|x_n-p\| \\
<{}& \big[\alpha_{n,0}+(1-\alpha_{n,0})\beta_{n,0}^1+(1-\alpha_{n,0})\big(1-\beta_{n,0}^1\big) \\
&+(1-\alpha_{n,0})\big(1-\beta_{n,0}^1\big)\big(1-\beta_{n,0}^2\big) \\
&+(1-\alpha_{n,0})\big(1-\beta_{n,0}^1\big)\big(1-\beta_{n,0}^2\big)\big(1-\beta_{n,0}^3\big)+\cdots \\
&+(1-\alpha_{n,0})\big(1-\beta_{n,0}^1\big)\big(1-\beta_{n,0}^2\big)\big(1-\beta_{n,0}^3\big)\times\cdots \\
&\times\big(1-\beta_{n,0}^{q-1}\big)\big(1-\beta_{n,0}^q\big)\big]\|x_n-p\|.
\end{aligned}
\tag{2.9}
$$

Using Lemma 1.3 in (2.9), we find the result that $\lim_{n\to\infty}\|x_{n+1}-p\|=0$.

That is $\{x_n\}_{n=0}^\infty$ converges strongly to p. This ends the proof. □

Theorem 2.2 leads to the following corollary.

Corollary 2.3 *Let $(E,\|\cdot\|)$ be a normed linear space, and let $T:E\to E$ be a self-map of E satisfying the contractive condition*

$$\big\|T^ix-T^iy\big\|\le a^i\|x-y\|+\sum_{j=0}^{i}\binom{i}{j}a^{i-j}\varphi\big(\|x-Tx\|\big), \tag{2.10}$$

for each $x,y\in E$, $0\le a<1$ and let $\varphi:R^+\to R^+$ be a subadditive monotone increasing function with $\varphi(0)=0$ and $\varphi(Lu)=L\varphi(u)$. Let $L\ge 0$, $u\in R^+$. For $x_0\in E$ then

(i) *T defined by* (2.10) *has a unique fixed point p;*
(ii) *the Kirk-Noor iterative scheme defined in* (1.10) *converges strongly to p of T;*
(iii) *the Kirk-Ishikawa iterative scheme defined in* (1.9) *converges strongly to p of T;*
(iv) *the Kirk-Mann iterative scheme defined in* (1.8) *converges strongly to p of T;*
(v) *the Kirk iterative scheme defined in* (1.7) *converges strongly to p of T.*

Theorem 2.4 *Let $(E,\|\cdot\|)$ be a normed linear space, and let $T:E\to E$ be a self-map of E satisfying the contractive condition*

$$\big\|T^ix-T^iy\big\|\le a^i\|x-y\|+\sum_{j=0}^{i}\binom{i}{j}a^{i-j}\varphi\big(\|x-Tx\|\big), \tag{2.11}$$

for each $x,y\in E$, $0\le a<1$ and let $\varphi:R^+\to R^+$ be a subadditive monotone increasing function with $\varphi(0)=0$ and $\varphi(Lu)=L\varphi(u)$. Let $L\ge 0$, $u\in R^+$. For $x_0\in E$, let $\{x_n\}_{n=0}^\infty$ be the Kirk-multistep-SP iterative scheme defined by (2.3). *Then*

(i) *T defined by* (2.11) *has a unique fixed point p;*
(ii) *the Kirk-multistep-SP iterative scheme converges strongly to p of T.*

Proof (i) We establish that the mapping T satisfying the contractive condition (2.11) has a unique fixed point.

Suppose there exist $p_1, p_2 \in F_T$, and that $p_1 \neq p_2$, with $\|p_1 - p_2\| > 0$, then

$$0 < \|p_1 - p_2\| = \|T^i p_1 - T^i p_2\| \leq \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi^j\bigl(\|p_1 - Tp_1\|\bigr) + a^i \|p_1 - p_2\|$$

$$= \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi^j(0) + a^i \|p_1 - p_2\|.$$

Thus,

$$\bigl(1 - a^i\bigr)\|p_1 - p_2\| \leq 0.$$

Since $a \in [0,1)$, then $1 - a^i > 0$ and $\|p_1 - p_2\| \leq 0$. Since the norm is nonnegative we have $\|p_1 - p_2\| = 0$. That is, $p_1 = p_2 = p$ (say). Thus, T has a unique fixed point p.

(ii) In view of (2.3) and (2.11), we have

$$\|x_{n+1} - p\| \leq \alpha_{n,0} \|y_n^1 - p\| + \sum_{i=1}^{k_1} \alpha_{n,i} \|T^i y_n^1 - Tp\|. \tag{2.12}$$

Using (2.11), with $y = y_n^1$, gives

$$\|Tp - T^i y_n^1\| \leq a^i \|y_n^1 - p\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\bigl(\|p - Tp\|\bigr). \tag{2.13}$$

Substituting (2.13) in (2.12), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_{n,0} \|y_n^1 - p\| + \left(\sum_{i=1}^{k_1} \alpha_{n,i} a^i\right) \|y_n^1 - p\| \\ &= \left(\sum_{i=0}^{k_1} \alpha_{n,i} a^i\right) \|y_n^1 - p\|. \end{aligned} \tag{2.14}$$

We note that $\beta_{n,i}^j \in [0,1]$ for each j and k_1, k_j are fixed integers (for each j), for $n = 1, 2, \ldots$ and $1 \leq j \leq q-1$. We have

$$\begin{aligned} \|y_n^1 - p\| &\leq \beta_{n,0}^1 \|y_n^2 - p\| + \sum_{i=1}^{k_2} \beta_{n,i}^1 \|T^i y_n^2 - Tp\| \\ &\leq \beta_{n,0}^1 \|y_n^2 - p\| + \sum_{i=1}^{k_2} \beta_{n,i}^1 a^i \|y_n^2 - p\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\bigl(\|p - Tp\|\bigr) \\ &= \beta_{n,0}^1 \|y_n^2 - p\| + \left(\sum_{i=1}^{k_2} \beta_{n,i}^1 a^i\right) \|y_n^2 - p\| \\ &= \left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right) \|y_n^2 - p\| \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right)\left[\beta_{n,0}^2 \|y_n^3 - p\| + \left(\sum_{i=1}^{k_3} \beta_{n,i}^2 a^i\right)\|y_n^3 - p\|\right] \\
&= \left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right)\left(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i\right)\|y_n^3 - p\| \\
&\leq \left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right)\left(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i\right)\left(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i\right)\|y_n^4 - p\| \\
&\leq \left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right)\left(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i\right)\left(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i\right) \times \cdots \\
&\quad \times \left(\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2} a^i\right)\left(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} a^i\right)\|x_n - p\|. \qquad (2.15)
\end{aligned}$$

Substituting (2.15) in (2.14), we have

$$\begin{aligned}
\|x_{n+1} - p\| \leq{} & \left(\sum_{i=0}^{k_1} \alpha_{n,i} a^i\right)\left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right)\left(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i\right)\left(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i\right) \times \cdots \\
& \times \left(\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2} a^i\right)\left(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} a^i\right)\|x_n - p\|. \qquad (2.16)
\end{aligned}$$

Since $a^i \in [0,1)$ and $\sum_{i=1}^{k_1} \alpha_{n,i} = \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j = 1$ for $j = 1, 2, 3, \ldots, q-1$, we have

$$\begin{aligned}
&\left(\sum_{i=0}^{k_1} \alpha_{n,i} a^i\right)\left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i\right)\left(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i\right)\left(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i\right) \cdots \left(\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2} a^i\right)\left(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1} a^i\right) \\
&\quad \leq \left(\sum_{i=0}^{k_1} \alpha_{n,i}\right)\left(\sum_{i=0}^{k_2} \beta_{n,i}^1\right)\left(\sum_{i=0}^{k_3} \beta_{n,i}^2\right)\left(\sum_{i=0}^{k_4} \beta_{n,i}^3\right) \times \cdots \\
&\qquad \times \left(\sum_{i=0}^{k_{q-1}} \beta_{n,i}^{q-2}\right)\left(\sum_{i=0}^{k_q} \beta_{n,i}^{q-1}\right) = 1. \qquad (2.17)
\end{aligned}$$

Using (2.17) and Lemma 1.3, (2.16) gives $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.

That is, $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof. □

Theorem 2.4 yields the following corollaries.

Corollary 2.5 *Let $(E, \|\cdot\|)$ be a normed linear space, and let $T : E \to E$ be a self-map of E satisfying the contractive condition*

$$\left\|T^i x - T^i y\right\| \leq a^i \|x - y\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\left(\|x - Tx\|\right), \qquad (2.18)$$

for each $x, y \in E$, $0 \leq a < 1$ and let $\varphi : R^+ \to R^+$ be a subadditive monotone increasing function with $\varphi(0) = 0$ and $\varphi(Lu) = L\varphi(u)$ $L \geq 0$, $u \in R^+$. For $x_0 \in E$ then

(i) *T defined by* (2.18) *has a unique fixed point p;*

(ii) *the Kirk-SP iterative scheme defined in* (1.11) *converges strongly to p of T;*

(iii) *the Kirk-Mann iterative scheme defined in* (1.8) *converges strongly to p of T*;
(iv) *the Kirk iterative scheme defined in* (1.7) *converges strongly to p of T*.

Example 2.6 Let $T : [0,1] \to [0,1] := \frac{x}{2}$,

$$\alpha_{n,0} = \beta^1_{n,0} = \beta^1_{n,1} = \alpha_{n,1} = \frac{4}{\sqrt{n}}, \quad n = 1,2,\dots,n_0 \text{ for some } n_0 \in \mathbf{N} \text{ and}$$

$$\alpha_{n,i} = 1 - \frac{8}{\sqrt{n}}, \quad \text{for } i = 2,3,\dots,k_1,$$

$$\beta_{n,i} = 1 - \frac{8}{\sqrt{n}}, \quad \text{for } i = 2,3,\dots,k_{j+1}, j = 2,3,\dots,q-2.$$

It is clear that T is a quasi-contractive operator satisfying (1.15) with a unique fixed point 0. Also it is easy to see that the above example satisfies all the conditions of Theorems 2.2 and 2.4.

3 Main result II

Some stability results for multistep-type iterative schemes in normed linear spaces

In this section, some stability results for the Kirk-multistep and Kirk-multistep-SP iterative schemes defined by (2.1) and (2.3) are established for contractive-type operators defined by (1.16). The stabilities of the Kirk-Noor, Kirk-Noor-SP, Kirk-Ishikawa, Kirk-Mann, and Kirk iterative schemes follow as corollaries.

Theorem 3.1 *Let $(E, \|\cdot\|)$ be a normed linear space, $T : E \to E$ be a self-map of E with a fixed point p satisfying the contractive condition*

$$\|Tx - Ty\| \leq a\|x - y\| + \varphi\big(\|x - Tx\|\big), \tag{3.1}$$

for each $x, y \in E$, $0 \leq a < 1$ and let $\varphi : R^+ \to R^+$ be a subadditive monotone increasing function with $\varphi(0) = 0$ and $\varphi(Lu) = L\varphi(u)$. Let $L \geq 0$, $u \in R^+$. For $x_0 \in E$, let $\{x_n\}_{n=0}^\infty$ be the Kirk-multistep-SP iterative scheme defined by (2.3). *Suppose T has a fixed point p. Then the Kirk-multistep-SP iterative scheme is T-stable.*

Proof Let $\{z_n\}_{n=0}^\infty$, $\{u^i_n\}_{n=0}^\infty$, for $i = 1,2,\dots,k-1$ be real sequences in E.

Let

$$\epsilon_n = \left\| z_{n+1} - \alpha_{n,0} u^1_n - \sum_{i=1}^{k_1} \alpha_{n,i} T^i u^1_n \right\|, \quad n = 0,1,2,\dots,$$

where

$$u^j_n = \beta^j_{n,0} u^{j+1}_n + \sum_{i=1}^{k_{j+1}} \beta^j_{n,i} T^i u^{j+1}_n, \quad \sum_{i=0}^{k_{j+1}} \beta^j_{n,i} = 1, j = 1,2,\dots,q-2,$$

$$u^{q-1}_n = \sum_{i=0}^{k_q} \beta^{q-1}_{n,i} T^i z_n, \quad \sum_{i=0}^{k_q} \beta^{q-1}_{n,i} = 1, q \geq 2$$

and let $\lim_{n\to\infty} \epsilon_n = 0$.

Then we shall prove that $\lim_{n\to\infty} z_n = p$ using the contractive mappings satisfying condition (3.1).

That is,

$$\begin{aligned}
\|z_{n+1} - p\| &\le \left\| z_{n+1} - \alpha_{n,0} u_n^1 - \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 \right\| + \left\| \alpha_{n,0} u_n^1 + \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 - p \right\| \\
&= \epsilon_n + \left\| \alpha_{n,0} u_n^1 + \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 - \sum_{i=0}^{k_1} \alpha_{n,i} T^i p \right\| \\
&= \epsilon_n + \left\| \alpha_{n,0} \bigl(u_n^1 - p\bigr) + \sum_{i=1}^{k_1} \alpha_{n,i} \bigl(T^i u_n^1 - T^i p\bigr) \right\| \\
&\le \epsilon_n + \alpha_{n,0} \bigl\| u_n^1 - p \bigr\| + \left(\sum_{i=1}^{k_1} \alpha_{n,i} \right) a^i \bigl\| u_n^1 - p \bigr\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\bigl(\|p - Tp\|\bigr) \\
&= \alpha_{n,0} \bigl\| u_n^1 - p \bigr\| + \left(\sum_{i=1}^{k_1} \alpha_{n,i} a^i \right) \bigl\| u_n^1 - p \bigr\| + \epsilon_n \\
&= \left(\sum_{i=0}^{k_1} \alpha_{n,i} a^i \right) \bigl\| u_n^1 - p \bigr\| + \epsilon_n.
\end{aligned} \tag{3.2}$$

Consider

$$\begin{aligned}
\bigl\| u_n^1 - p \bigr\| &= \left\| \beta_{n,0} u_n^2 + \sum_{i=1}^{k_2} \beta_{n,i} T^i u_n^2 - \sum_{i=0}^{k_2} \beta_{n,i} T^i p \right\| \\
&= \left\| \beta_{n,0} \bigl(u_n^2 - p\bigr) + \sum_{i=1}^{k_2} \beta_{n,i} \bigl(T^i u_n^2 - T^i p\bigr) \right\| \\
&\le \beta_{n,0} \bigl\| u_n^2 - p \bigr\| + \left(\sum_{i=1}^{k_2} \beta_{n,i} \right) \left[a^i \bigl\| u_n^2 - p \bigr\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\bigl(\|p - Tp\|\bigr) \right] \\
&= \beta_{n,0} \bigl\| u_n^2 - p \bigr\| + \left(\sum_{i=1}^{k_2} \beta_{n,i} a^i \right) \bigl\| u_n^2 - p \bigr\| \\
&= \left(\sum_{i=0}^{k_2} \beta_{n,i} a^i \right) \bigl\| u_n^2 - p \bigr\| \\
&\le \left(\sum_{i=0}^{k_2} \beta_{n,i} a^i \right) \left[\beta_{n,0} \bigl\| u_n^3 - p \bigr\| + \left(\sum_{i=1}^{k_3} \beta_{n,i} \right) \left[a^i \bigl\| u_n^3 - p \bigr\| \right.\right. \\
&\quad \left.\left. + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\bigl(\|p - Tp\|\bigr) \right] \right] \\
&= \left(\sum_{i=0}^{k_2} \beta_{n,i} a^i \right) \left(\sum_{i=0}^{k_3} \beta_{n,i} a^i \right) \bigl\| u_n^3 - p \bigr\| \\
&\le \left(\sum_{i=0}^{k_2} \beta_{n,i}^1 a^i \right) \left(\sum_{i=0}^{k_3} \beta_{n,i}^2 a^i \right) \left(\sum_{i=0}^{k_4} \beta_{n,i}^3 a^i \right) \bigl\| u_n^4 - p \bigr\|
\end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{i=0}^{k_2}\beta^1_{n,i}a^i\right)\left(\sum_{i=0}^{k_3}\beta^2_{n,i}a^i\right)\left(\sum_{i=0}^{k_4}\beta^3_{n,i}a^i\right)\times\cdots \\ &\quad\times\left(\sum_{i=0}^{k_{q-1}}\beta^{q-2}_{n,i}a^i\right)\left(\sum_{i=0}^{k_q}\beta^{q-1}_{n,i}a^i\right)\|z_n-p\|; \end{aligned} \tag{3.3}$$

(3.3) holds, since $Tp = p$ and $\varphi(0) = 0$.

Substituting (3.3) in (3.2), we have

$$\begin{aligned} \|z_{n+1}-p\| &\leq \left(\sum_{i=0}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=0}^{k_2}\beta^1_{n,i}a^i\right)\left(\sum_{i=0}^{k_3}\beta^2_{n,i}a^i\right)\left(\sum_{i=0}^{k_4}\beta^3_{n,i}a^i\right)\times\cdots \\ &\quad\times\left(\sum_{i=0}^{k_{q-1}}\beta^{q-2}_{n,i}a^i\right)\left(\sum_{i=0}^{k_q}\beta^{q-1}_{n,i}a^i\right)\|z_n-p\|+\epsilon_n. \end{aligned} \tag{3.4}$$

Since $a^i \in [0,1)$ and $\sum_{i=1}^{k_1}\alpha_{n,i} = \sum_{i=1}^{k_{j+1}}\beta^j_{n,i} = 1$ for $j = 1,2,3,\ldots,q-1$, and

$$\begin{aligned} &\left(\sum_{i=0}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=0}^{k_2}\beta^1_{n,i}a^i\right)\left(\sum_{i=0}^{k_3}\beta^2_{n,i}a^i\right)\left(\sum_{i=0}^{k_4}\beta^3_{n,i}a^i\right)\cdots\left(\sum_{i=0}^{k_{q-1}}\beta^{q-2}_{n,i}a^i\right)\left(\sum_{i=0}^{k_q}\beta^{q-1}_{n,i}a^i\right) \\ &\quad< \left(\sum_{i=0}^{k_1}\alpha_{n,i}\right)\left(\sum_{i=0}^{k_2}\beta^1_{n,i}\right)\left(\sum_{i=0}^{k_3}\beta^2_{n,i}\right)\left(\sum_{i=0}^{k_4}\beta^3_{n,i}\right)\cdots\left(\sum_{i=0}^{k_{q-1}}\beta^{q-2}_{n,i}\right)\left(\sum_{i=0}^{k_q}\beta^{q-1}_{n,i}\right)=1. \end{aligned} \tag{3.5}$$

Let

$$\delta = \left(\sum_{i=0}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=0}^{k_2}\beta^1_{n,i}a^i\right)\left(\sum_{i=0}^{k_3}\beta^2_{n,i}a^i\right)\left(\sum_{i=0}^{k_4}\beta^3_{n,i}a^i\right)\cdots\left(\sum_{i=0}^{k_{q-1}}\beta^{q-2}_{n,i}a^i\right)\left(\sum_{i=0}^{k_q}\beta^{q-1}_{n,i}a^i\right)$$

then $\delta < 1$. Hence

$$\|z_{n+1}-p\| \leq \delta\|z_n-p\|+\epsilon_n. \tag{3.6}$$

Using Lemma 1.3 in (3.6), we have

$$\lim_{n\to\infty} z_n = p.$$

Conversely, let $\lim_{n\to\infty} z_n = p$, and we show that $\lim_{n\to\infty}\epsilon_n = 0$ as follows:

$$\begin{aligned} \epsilon_n &= \left\|z_{n+1}-\alpha_{n,0}u^1_n-\sum_{i=0}^{k_1}\alpha_{n,i}T^iu^1_n\right\| \leq \|z_{n+1}-p\|+\left\|p-\alpha_{n,0}u^1_n-\sum_{i=0}^{k_1}\alpha_{n,i}T^iu^1_n\right\| \\ &= \|z_{n+1}-p\|+\left\|\sum_{i=0}^{k_1}\alpha_{n,i}T^ip-\alpha_{n,0}u^1_n-\sum_{i=0}^{k_1}\alpha_{n,i}T^iu^1_n\right\| \\ &= \|z_{n+1}-p\|+\left\|\alpha_{n,0}\left(u^1_n-p\right)+\sum_{i=1}^{k_1}\alpha_{n,i}\left(T^ip-T^iu^1_n\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&\le \|z_{n+1}-p\| + \alpha_{n,0}\|u_n^1 - p\| + \sum_{i=1}^{k_1}\alpha_{n,i}\|T^i p - T^i u_n^1\| \\
&\le \|z_{n+1}-p\| + \alpha_{n,0}\|u_n^1 - p\| + \left(\sum_{i=1}^{k_1}\alpha_{n,i}\right)\left[a^i\|u_n^1 - p\| + \sum_{j=0}^{i}\binom{i}{j}a^{i-j}\varphi(\|p-Tp\|)\right] \\
&= \|z_{n+1}-p\| + \alpha_{n,0}\|u_n^1 - p\| + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\|u_n^1 - p\| \\
&= \|z_{n+1}-p\| + \left(\sum_{i=0}^{k_1}\alpha_{n,i}a^i\right)\|u_n^1 - p\|.
\end{aligned}
\tag{3.7}
$$

Substituting $\|u_n^1 - p\|$, that is, (3.3) in (3.7), we have

$$
\begin{aligned}
\epsilon_n \le{}& \|z_{n+1}-p\| + \left(\sum_{i=0}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=0}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=0}^{k_2}\beta_{n,i}^1 a^i\right)\left(\sum_{i=0}^{k_3}\beta_{n,i}^2 a^i\right)\left(\sum_{i=0}^{k_4}\beta_{n,i}^3 a^i\right)\times\cdots \\
&\times\left(\sum_{i=0}^{k_{q-1}}\beta_{n,i}^{q-2}a^i\right)\left(\sum_{i=0}^{k_q}\beta_{n,i}^{q-1}a^i\right)\|z_n - p\|.
\end{aligned}
\tag{3.8}
$$

Using (3.5), (3.8) becomes

$$
\epsilon_n \le \|z_{n+1}-p\| + \delta\|z_n - p\|.
$$

Hence, using $\lim_{n\to\infty}\|z_n - p\| = 0$ (by our assumption).

We have $\lim_{n\to\infty}\epsilon_n = 0$.

Therefore the Kirk-multistep-SP iterative scheme (2.3) is T-stable. This ends the proof. □

Theorem 3.1 yields the following corollary.

Corollary 3.2 *Let $(E,\|\cdot\|)$ be a normed linear space, $T: E\to E$ be a self-map of E with a fixed point p satisfying the contractive condition*

$$
\|Tx - Ty\| \le a\|x-y\| + \varphi(\|x - Tx\|),
\tag{3.9}
$$

for each $x,y\in E$, $0\le a<1$ and let $\varphi: R^+\to R^+$ be a subadditive monotone increasing function with $\varphi(0)=0$ and $\varphi(Lu)=L\varphi(u)$. Let $L\ge 0$, $u\in R^+$. For $x_0\in E$, let $\{x_n\}_{n=0}^\infty$ be the Kirk-SP, Kirk-Mann, and Kirk iterative schemes defined by (1.11), (1.8), *and* (1.7), *respectively. Suppose T has a fixed point p. Then*

(i) *the Kirk-SP iterative scheme* (1.11) *is T-stable*;
(ii) *the Kirk-Mann iterative scheme* (1.8) *is T-stable*;
(iii) *the Kirk iterative scheme* (1.7) *is T-stable*.

Theorem 3.3 *Let $(E,\|\cdot\|)$ be a normed linear space, $T: E\to E$ be a self-map of E with a fixed point p satisfying the contractive condition*

$$
\|Tx - Ty\| \le a\|x-y\| + \varphi(\|x - Tx\|),
\tag{3.10}
$$

for each $x, y \in E$, $0 \leq a < 1$ and let $\varphi : R^+ \to R^+$ be a subadditive monotone increasing function with $\varphi(0) = 0$ and $\varphi(Lu) = L\varphi(u)$. Let $L \geq 0$, $u \in R^+$. For $x_0 \in E$, let $\{x_n\}_{n=0}^\infty$ be the Kirk-multistep iterative scheme defined by (2.1). *Suppose T has a fixed point p. Then the Kirk-multistep iterative scheme is T-stable.*

Proof Let $\{z_n\}_{n=0}^\infty$, $\{u_n^i\}_{n=0}^\infty$, for $i = 1, 2, \ldots, k-1$ be real sequences in E.

Let

$$\epsilon_n = \left\| z_{n+1} - \alpha_{n,0} z_n - \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 \right\|, \quad n = 0, 1, 2, \ldots,$$

where

$$u_n^j = \beta_{n,0}^j z_n + \sum_{i=1}^{k_{j+1}} \beta_{n,i}^j T^i u_n^{j+1}, \quad \sum_{i=0}^{k_{j+1}} \beta_{n,i}^j = 1, j = 1, 2, \ldots, q-2,$$

$$u_n^{q-1} = \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} T^i z_n, \quad \sum_{i=0}^{k_q} \beta_{n,i}^{q-1} = 1, \quad q \geq 2$$

and let $\lim_{n\to\infty} \epsilon_n = 0$.

Then we shall prove that $\lim_{n\to\infty} z_n = p$ using the contractive mappings satisfying condition (3.10).

That is,

$$\begin{aligned} \|z_{n+1} - p\| &\leq \left\| z_{n+1} - \alpha_{n,0} z_n - \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 \right\| + \left\| \alpha_{n,0} z_n + \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 - p \right\| \\ &= \epsilon_n + \left\| \alpha_{n,0} z_n + \sum_{i=1}^{k_1} \alpha_{n,i} T^i u_n^1 - \sum_{i=0}^{k_1} \alpha_{n,i} T^i p \right\| \\ &= \epsilon_n + \left\| \alpha_{n,0}(z_n - p) + \sum_{i=1}^{k_1} \alpha_{n,i} \left(T^i u_n^1 - T^i p \right) \right\| \\ &\leq \epsilon_n + \alpha_{n,0} \|z_n - p\| + \left(\sum_{i=1}^{k_1} \alpha_{n,i} \right) a^i \left\| u_n^1 - p \right\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\left(\|p - Tp\| \right) \\ &= \alpha_{n,0} \|z_n - p\| + \left(\sum_{i=1}^{k_1} \alpha_{n,i} a^i \right) \left\| u_n^1 - p \right\| + \epsilon_n. \end{aligned} \tag{3.11}$$

Consider

$$\begin{aligned} \left\| u_n^1 - p \right\| &= \left\| \beta_{n,0} z_n + \sum_{i=1}^{k_2} \beta_{n,i} T^i u_n^2 - \sum_{i=0}^{k_2} \beta_{n,i} T^i p \right\| \\ &= \left\| \beta_{n,0}(z_n - p) + \sum_{i=1}^{k_2} \beta_{n,i} \left(T^i u_n^2 - T^i p \right) \right\| \\ &\leq \beta_{n,0} \|z_n - p\| + \left(\sum_{i=1}^{k_2} \beta_{n,i} \right) \left[a^i \left\| u_n^2 - p \right\| + \sum_{j=0}^{i} \binom{i}{j} a^{i-j} \varphi\left(\|p - Tp\| \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \beta_{n,0}\|z_n - p\| + \left(\sum_{i=1}^{k_2}\beta_{n,i}a^i\right)\left\|u_n^2 - p\right\| \\
&\leq \beta_{n,0}\|z_n - p\| + \left(\sum_{i=1}^{k_2}\beta_{n,i}a^i\right)\left[\beta_{n,0}\|z_n - p\| + \left(\sum_{i=1}^{k_3}\beta_{n,i}\right)\left[a^i\left\|u_n^3 - p\right\|\right.\right. \\
&\quad \left.\left. + \sum_{j=0}^{i}\binom{i}{j}a^{i-j}\varphi\left(\|p - Tp\|\right)\right]\right] \\
&= \beta_{n,0}\|z_n - p\| + \left(\sum_{i=1}^{k_2}\beta_{n,i}a^i\right)\beta_{n,0}\|z_n - p\| \\
&\quad + \left(\sum_{i=1}^{k_2}\beta_{n,i}a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}\right)a^i\left\|u_n^3 - p\right\| \\
&\leq \beta_{n,0}^1\|z_n - p\| + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\beta_{n,0}^2\|z_n - p\| \\
&\quad + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\beta_{n,0}^3\|z_n - p\| \\
&\quad + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\left\|y_n^4 - p\right\| \\
&\leq \beta_{n,0}^1\|z_n - p\| + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\beta_{n,0}^2\|z_n - p\| \\
&\quad + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\beta_{n,0}^3\|z_n - p\| \\
&\quad + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\beta_{n,0}^4\|z_n - p\| + \cdots \\
&\quad + \left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right) \times \cdots \\
&\quad \times \left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2}a^i\right)\left(\sum_{i=1}^{k_q}\beta_{n,i}^{q-1}a^i\right)\beta_{n,0}^q\|z_n - p\|;
\end{aligned}
\tag{3.12}
$$

(3.12) holds, since $Tp = p$ and $\varphi(0) = 0$.

Substituting (3.12) in (3.11), we have

$$
\begin{aligned}
\|z_{n+1} - p\| \leq{}& \alpha_{n,0}\|z_n - p\| + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\beta_{n,0}^1\|z_n - p\| \\
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\beta_{n,0}^2\|z_n - p\| \\
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\beta_{n,0}^3\|z_n - p\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\beta_{n,0}^4\|z_n-p\| + \cdots \\
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\times\cdots \\
&\times\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2}a^i\right)\left(\sum_{i=1}^{k_q}\beta_{n,i}^{q-1}a^i\right)\beta_{n,0}^q\|z_n-p\| + \epsilon_n \\
= &\left[\alpha_{n,0} + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\beta_{n,0}^1 + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\beta_{n,0}^2\right. \\
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\beta_{n,0}^3 \\
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\beta_{n,0}^4\|z_n-p\| + \cdots \\
&+ \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\times\cdots \\
&\left.\times\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2}a^i\right)\left(\sum_{i=1}^{k_q}\beta_{n,i}^{q-1}a^i\right)\beta_{n,0}^q\right]\|z_n-p\| + \epsilon_n \\
< &\left[\alpha_{n,0} + (1-\alpha_{n,0})\beta_{n,0}^1 + (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right) + (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\left(1-\beta_{n,0}^2\right)\right. \\
&+ (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\left(1-\beta_{n,0}^2\right)\left(1-\beta_{n,0}^3\right) + \cdots \\
&+ (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\left(1-\beta_{n,0}^2\right)\left(1-\beta_{n,0}^3\right)\times\cdots \\
&\left.\times\left(1-\beta_{n,0}^{q-1}\right)\left(1-\beta_{n,0}^q\right)\right]\|z_n-p\| + \epsilon_n.
\end{aligned}
\tag{3.13}
$$

Using Lemma 1.3 in (3.13), we have

$$\lim_{n\to\infty} z_n = p.$$

Conversely, let $\lim_{n\to\infty} z_n = p$, we show that $\lim_{n\to\infty}\epsilon_n = 0$ as follows:

$$
\begin{aligned}
\epsilon_n &= \left\|z_{n+1} - \alpha_{n,0}z_n - \sum_{i=0}^{k_1}\alpha_{n,i}T^iu_n^1\right\| \\
&\le \|z_{n+1}-p\| + \left\|p - \alpha_{n,0}z_n - \sum_{i=0}^{k_1}\alpha_{n,i}T^iu_n^1\right\| \\
&= \|z_{n+1}-p\| + \left\|\sum_{i=0}^{k_1}\alpha_{n,i}T^ip - \alpha_{n,0}z_n - \sum_{i=0}^{k_1}\alpha_{n,i}T^iu_n^1\right\| \\
&= \|z_{n+1}-p\| + \left\|\alpha_{n,0}(z_n-p) + \sum_{i=1}^{k_1}\alpha_{n,i}\left(T^ip - T^iu_n^1\right)\right\| \\
&\le \|z_{n+1}-p\| + \alpha_{n,0}\|z_n-p\| + \sum_{i=1}^{k_1}\alpha_{n,i}\left\|T^ip - T^iu_n^1\right\|
\end{aligned}
$$

$$\begin{aligned}
&\le \|z_{n+1}-p\| + \alpha_{n,0}\|z_n-p\| \\
&\quad + \left(\sum_{i=1}^{k_1}\alpha_{n,i}\right)\left[a^i\left\|u_n^1-p\right\| + \sum_{j=0}^{i}\binom{i}{j}a^{i-j}\varphi\left(\|p-Tp\|\right)\right] \\
&= \|z_{n+1}-p\| + \alpha_{n,0}\|z_n-p\| + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left\|u_n^1-p\right\|. \qquad (3.14)
\end{aligned}$$

Substituting $\|u_n^1-p\|$, that is, (3.12) in (3.14), we have

$$\begin{aligned}
\epsilon_n &\le \|z_{n+1}-p\| + \alpha_{n,0}\|z_n-p\| \\
&\quad + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\beta_{n,0}^1\|z_n-p\| + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\beta_{n,0}^2\|z_n-p\| \\
&\quad + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\beta_{n,0}^3\|z_n-p\| \\
&\quad + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\beta_{n,0}^4\|z_n-p\| + \cdots \\
&\quad + \left(\sum_{i=1}^{k_1}\alpha_{n,i}a^i\right)\left(\sum_{i=1}^{k_2}\beta_{n,i}^1a^i\right)\left(\sum_{i=1}^{k_3}\beta_{n,i}^2a^i\right)\left(\sum_{i=1}^{k_4}\beta_{n,i}^3a^i\right)\times\cdots \\
&\quad \times\left(\sum_{i=1}^{k_{q-1}}\beta_{n,i}^{q-2}a^i\right)\left(\sum_{i=1}^{k_q}\beta_{n,i}^{q-1}a^i\right)\beta_{n,0}^q\|z_n-p\| \\
&\le \|z_{n+1}-p\| + \left[\alpha_{n,0} + (1-\alpha_{n,0})\beta_{n,0}^1 + (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\right. \\
&\quad + (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\left(1-\beta_{n,0}^2\right) \\
&\quad + (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\left(1-\beta_{n,0}^2\right)\left(1-\beta_{n,0}^3\right) + \cdots \\
&\quad + (1-\alpha_{n,0})\left(1-\beta_{n,0}^1\right)\left(1-\beta_{n,0}^2\right)\left(1-\beta_{n,0}^3\right)\times\cdots \\
&\quad \left.\times\left(1-\beta_{n,0}^{q-1}\right)\left(1-\beta_{n,0}^q\right)\right]\|z_n-p\|. \qquad (3.15)
\end{aligned}$$

Hence, using $\lim_{n\to\infty}\|z_n-p\| = 0$ (by our assumption).

We have $\lim_{n\to\infty}\epsilon_n = 0$.

Therefore the Kirk-multistep iterative scheme (2.1) is T-stable. This ends the proof. □

Theorem 3.3 yields the following corollary.

Corollary 3.4 *Let $(E,\|\cdot\|)$ be a normed linear space, $T: E\to E$ be a self-map of E with a fixed point p satisfying the contractive condition*

$$\|Tx-Ty\| \le a\|x-y\| + \varphi\left(\|x-Tx\|\right), \qquad (3.16)$$

for each $x,y\in E$, $0\le a<1$ and let $\varphi: R^+\to R^+$ be a subadditive monotone increasing function with $\varphi(0)=0$ and $\varphi(Lu)=L\varphi(u)$ $L\ge 0$, $u\in R^+$. For $x_0\in E$, let $\{x_n\}_{n=0}^\infty$ be the Kirk-Noor, *Kirk-Ishikawa, Kirk-Mann, and Kirk iterative schemes defined by* (1.10), (1.9), (1.8), *and* (1.7), *respectively. Suppose T has a fixed point p. Then*

(i) *the Kirk-Noor iterative scheme* (1.10) *is T-stable*;
(ii) *the Kirk-Ishikawa iterative scheme* (1.9) *is T-stable*;
(iii) *the Kirk-Mann iterative scheme* (1.8) *is T-stable*;
(iv) *the Kirk iterative scheme* (1.7) *is T-stable*.

4 Numerical examples

In this section, we use some examples to compare our modified iterative schemes (Kirk-multstep-SP hybrid iterative schemes) with the Kirk-SP scheme with the help of computer programs in PYTHON 2.5.4. The results are shown in Tables 1 to 5. We take $\alpha_{n,1} = \beta_{n,1} = \gamma_{n,1} = \beta_n^j = \frac{1}{(C+n)^{\frac{1}{2}}}$, where $0 < C \leq 1$ (for $j = 1, 2, 3, \ldots, k-2$) $\alpha_{n,0} = 1 - \sum_{i=0}^{k_1} \alpha_{n,i}$, $\beta_{n,0}^1 = \sum_{i=0}^{k_2} \beta_{n,i}^1 \cdots \beta_{n,0}^{q-1} = \sum_{i=0}^{k_q} \beta_{n,i}^{q-1}$ for all the iterative schemes.

4.1 Example of increasing function

Let $f : [0,2] \to [0,2]$ be defined by $f(x) = (\frac{\pi + x_n(4-x_n^2)^{\frac{1}{2}} - (4-2x_n^2)\sin^-1(x_n/2)}{\pi})^{\frac{1}{2}}$. Then f is an increasing function. The comparison of these iterative schemes to the fixed point $p = 1.00000000$ is shown in Table 1.

4.2 Example of decreasing function

Let $f : [0,1] \to [0,1]$ be defined by $f(x) = (1-x^3)^{\frac{1}{2}}$, a decreasing function. By taking initial approximation $x_0 = 0.8$, the comparison of these iterative schemes to the fixed point $p = 0.18834768$ is shown in Table 2.

4.3 Example of functions with multiple zero

The function defined by $f(x) = (1-x)^2$ is a function with multiple zeros. By taking the initial approximation $x_0 = 0.9$, the comparison of the convergence of these iterative schemes to the exact fixed point $p = 0.38196601$ is shown in Table 3.

Table 1 Numerical example for increasing functions

n	Kirk-SP	Kirk-multistep-SP
0	0.80000000	0.80000000
1	1.00048052	1.00000000
2	1.00000328	1.00000000
3	1.00000002	1.00000000
4	1.00000000	1.00000000
5	1.00000000	1.00000000

Table 2 Numerical example for decreasing functions

n	Kirk-SP	Kirk-multistep-SP
0	0.80000000	0.80000000
1	0.18829774	0.18834768
2	0.18834722	0.18834768
3	0.18834768	0.18834768
4	0.18834768	0.18834768
5	0.18834768	0.18834768

Table 3 Numerical example for functions with multiple zero

n	Kirk-SP	Kirk-multistep-SP
0	0.90000000	0.90000000
1	0.38248950	0.38196601
2	0.38196611	0.38196601
3	0.38196601	0.38196601
4	0.38196601	0.38196601
5	0.38196601	0.38196601

Table 4 Numerical example for cubic equation

n	Kirk-SP	Kirk-multistep-SP
0	0.80000000	0.80000000
1	0.75463303	0.75487767
2	0.75487802	0.75487767
3	0.75487767	0.75487767
4	0.75487767	0.75487767
5	0.75487767	0.75487767

Table 5 Numerical example for oscillatory functions

n	Kirk-SP	Kirk-multistep-SP
0	4.00000000	4.00000000
1	1.14558649	1.00000000
2	1.00000002	1.00000000
3	1.00000000	1.00000000
4	1.00000000	1.00000000
5	1.00000000	1.00000000

4.4 Example of cubic equation

To find the root of the equation $x^3 + x^2 - 1 = 0$ means to find the fixed point of the function $(1 - x^3)^{\frac{1}{2}}$ as $x^3 + x^2 - 1 = 0$, which can be rewritten as $(1 - x^3)^{\frac{1}{2}} = x$. The comparison of the convergence of these various iterative schemes to the exact fixed point $p = 0.75487767$ of $(1 - x^3)^{\frac{1}{2}}$ is shown in Table 4.

4.5 Example of oscillatory functions

The function defined by $f(x) = \frac{1}{x}$ is an oscillatory function. By taking initial approximation $x_0 = 4$, the comparison of the convergence of these iterative schemes to the exact fixed point $p = 1.00000000$ is shown in Tables 5.

5 Observations

5.1 Increasing function $f(x) = (\frac{\pi + x_n(4 - x_n^2)^{\frac{1}{2}} - (4 - 2x_n^2)\sin^{-1}(x_n/2)}{\pi})^{\frac{1}{2}}$

The Kirk-SP iterative scheme converges to a fixed point in four iterations while the Kirk-multistep-SP hybrid scheme converges in one iteration.

5.2 Decreasing function $f(x) = (1 - x^3)^{\frac{1}{2}}$

The Kirk-SP iterative scheme converges to a fixed point in three iterations while the Kirk-multistep-SP hybrid scheme converges in one iteration.

5.3 Function with multiple zero $f(x) = (1 - x)^2$

The Kirk-SP iterative scheme converges to a fixed point in three iterations while the Kirk-multistep-SP hybrid scheme converges in one iteration.

5.4 Cubic equation $x^3 + x^2 - 1 = 0$

The Kirk-SP iterative scheme converges to a fixed point in three iterations while the Kirk-multistep-SP hybrid scheme converges in one iteration.

5.5 Oscillatory functions $f(x) = \frac{1}{x}$

The Kirk-SP iterative scheme converges to a fixed point in three iterations while the Kirk-multistep-SP hybrid scheme converges in one iteration.

5.6 Remark

Our Kirk-multistep-SP hybrid iterative schemes converge faster than the Kirk-SP iterative scheme for increasing, decreasing, function with multiple zero, cubic equation, and oscillatory function.

6 Conclusion

Our Kirk-multistep-SP hybrid iterative schemes have good potentials for further applications.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first two authors contributed equally and significantly in this research work, while the third author proof-read the manuscript. All authors read and approved the final manuscript.

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