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Some new coupled fixed point theorems on partial metric spaces

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ABSTRACT

In this paper, we obtain some new coupled fixed point theorems for mappings satisfying some contractive conditions on complete partial metric space. Our results unify, extend and generalize the results of [3] and [12].

Keywords: Metric spaces, partial metric spaces, complete partial metric space, coupled fixed point, unique coupled fixed point..

1. Introduction

The concept of coupled fixed point for a partially ordered set X was introduced in [4] by Bhaskar and Lakshmikantham. Several other authors such as Ciric and Lakshmikantham [5], Sabetghadam *et al.* [12] have proved some coupled fixed point results in metric spaces. The concept of a partial metric space (PMS) was introduced in 1992 by Matthews [7]. The PMS is a generalization of the usual metric spaces in which $d(x,x)$ need not be zero. Recently, many authors have contributed much to the literature (see [1-3,6-7,8-11]). Recently, Hassen Aydi [3] proved some coupled fixed point results on PMS. Our new results are unification, an extension and a generalization of [3] and [12]. In the sequel, we give some definitions of some applicable concepts.

Definition 1.1 [3]. An element $(x,y)\in X\times X$ is said to be a coupled fixed point of the mapping $F: X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Definition 1.2 [3]. A partial metric on a nonempty set X is a function $p: X\times X\to\Re^+$ such that for all $x,y,z\in X$:

(p1) $x=y\Leftrightarrow p(x,x)=p(x,y)=p(y,y)$

(p2) $p(x,y)\le p(x,y)$

(p3) $p(x,y)=p(y,x)$

(p4) $p(x,y)\le p(x,z)+p(z,y)-p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark 1.1. Observe that if $p(x,y)=0$, then by (p1), (p2) and (p3) $x=y$. But $x=y$ does not imply that $p(x,y)$ is zero.

If p is a partial metric on a nonempty set X, then the function $p^s: X\times X\to\Re_+$ given by $p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$, is a metric on X.

Example 1.1 [6]. If $X:=\Re^{N_0}\cup\bigcup_{n\ge1}\Re^{\{0,1,\cdots,n-1\}}$, where N_0 is the setof nonnegative integers. By $L(x)$ denote the set $\{0,1,\cdots,n\}$ if $x\in\Re^{\{0,1,\cdots,n-1\}}$ for some $n\in N$, and the set N_0 if $x\in\Re^{N_0}$. Then a partial metric is defined on X by

$$p(x,y)=\inf\{2^{-i}|i\in L(x)\cap L(y)\quad and\quad \forall j\in N_0(j<i\Rightarrow x(j)=y(j))\}.$$

Set $\rho_p:=\inf\{p(x,y): x,y\in X\}=\inf\{p(x,x): x\in X\}$. Notice that X_p may be empty.

Example 1.2 [6]. If $X:=\{[a,b]|a,b\in\Re, a\le b\}$ then $p([a,b],[c,d])=\max\{b,d\}-\min\{a,b\}$ defines a partial metric p on X.

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Definition 1.3 [3]. Let (X,p) be a partial metric space, then

(i) a sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$;

(ii) a sequence $\{x_n\}$ in a partial metric space (X,P) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$;

(iii) a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is,

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Lemma 1.1 [3]. Let (X,p) be a partial metric space;

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) a partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete; furthermore, $\lim_{n\to+\infty} p^s(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Hassen Aydi [3] proved the following coupled fixed point results on PMS.

$$p\big(F(x,y),F(u,v)\big) \le kp(x,u) + lp(y,v) \tag{1}$$

$$p(F(x,y),F(u,v)) \le kp(F(x,y),x) + lp(F(u,v),u) \tag{2}$$

$$p(F(x,y),F(u,v)) \le kp(F(x,y),u) + lp(F(u,v),x) \tag{3}$$

with $k+l<1$ in (1.1) - (1.2) and $k+2l<1$ in (3).

2. Main Results

Theorem 2.1. Let (X,p) be a complete partial metric space. Suppose that the mapping $F: X\times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$

$$\begin{aligned} p(F(x,y),F(u,v)) \quad \le \quad & a_1p(x,u) + a_2p(y,v) + a_3p(F(x,y),x) \\ & + a_4p(F(u,v),u) + a_5p(F(x,y),u) \\ & + a_6p(F(u,v),x) \end{aligned} \tag{4}$$

where $a_1, a_2, \cdots, a_6$ are nonnegative constants with $a_1+a_2+a_3+a_4+a_5+2a_6<1$. Then, F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Continuing this process, set $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$. Then by (4), we obtain:

$$\begin{aligned} p(x_n,x_{n+1}) &= p(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ &\le a_1p(x_{n-1},x_n) + a_2p(y_{n-1},y_n) + a_3p(F(x_{n-1},y_{n-1}),x_{n-1}) \\ &\quad + a_4p(F(x_n,y_n),x_n) + a_5p(F(x_{n-1},y_{n-1}),x_n) \\ &\quad + a_6p(F(x_n,y_n),x_{n-1}) \\ &= a_1p(x_{n-1},x_n) + a_2p(y_{n-1},y_n) + a_3p(x_n,x_{n-1}) \\ &\quad + a_4p(x_{n+1},x_n) + a_5p(x_n,x_n) + a_6p(x_{n+1},x_{n-1}) \\ &\le a_1p(x_{n-1},x_n) + a_2p(y_{n-1},y_n) + a_3p(x_n,x_{n-1}) \\ &\quad + a_4p(x_{n+1},x_n) + a_5p(x_n,x_{n+1}) + a_6p(x_{n+1},x_{n-1}) \\ &\le a_1p(x_{n-1},x_n) + a_2p(y_{n-1},y_n) + a_3p(x_n,x_{n-1}) \\ &\quad + a_4p(x_{n+1},x_n) + a_5p(x_n,x_{n+1}) \\ &\quad + a_6p(x_{n+1},x_n) + a_6p(x_n,x_{n-1}). \end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned} p(y_n,y_{n+1}) \quad \le \quad & a_1p(y_{n-1},y_n) + a_2p(x_{n-1},x_n) + a_3p(y_n,y_{n-1}) \\ & + a_4p(y_{n+1},y_n) + a_5p(y_n,y_{n+1}) + a_6p(y_{n+1},y_n) \\ & + a_6p(y_n,y_{n-1}). \end{aligned} \tag{6}$$

Set

$$\begin{aligned} d_n &= p(x_n,x_{n+1}) + p(y_n,y_{n+1}) \\ &\le a_1p(x_{n-1},x_n) + a_2p(y_{n-1},y_n) + a_3p(x_n,x_{n-1}) \\ &\quad + a_4p(x_{n+1},x_n) + a_5p(x_n,x_{n+1}) + a_6p(x_{n+1},x_n) + \\ &\quad a_6p(x_n,x_{n-1}) + a_1p(y_{n-1},y_n) + a_2p(x_{n-1},x_n) \\ &\quad + a_3p(y_n,y_{n-1}) + a_4p(y_{n+1},y_n) + a_5p(y_n,y_{n+1}) \\ &\quad + a_6p(y_{n+1},y_n) + a_6p(y_n,y_{n-1}) \\ &= (a_1+a_2+a_3+a_6)p(x_n,x_{n-1}) \\ &\quad + (a_1+a_2+a_3+a_6)p(y_n,y_{n-1}) \\ &\quad + (a_4+a_5+a_6)p(x_n,x_{n+1}) \\ &\quad + (a_4+a_5+a_6)p(y_n,y_{n+1}). \end{aligned} \tag{7}$$

Hence:

$$
\begin{aligned}
(1-a_4-a_5-a_6)[p(x_n,x_{n+1}) \quad + \quad & p(y_n,y_{n+1})] \\
& \leq (a_1+a_2+a_3+a_6) \\
& p(x_n,x_{n-1})+(a_1+a_2 \\
& +a_3+a_6)p(y_n,y_{n-1}).
\end{aligned} \tag{8}
$$

This implies that

$$
\begin{aligned}
p(x_n,x_{n+1})+p(y_n,y_{n+1}) \quad \leq \quad & \frac{a_1+a_2+a_3+a_6}{1-a_4-a_5-a_6}[p(x_n,x_{n-1}) \\
& +p(y_n,y_{n-1})].
\end{aligned} \tag{9}
$$

i.e $d_n \leq \lambda d_{n-1}$, where $\lambda = \frac{a_1+a_2+a_3+a_6}{1-a_4-a_5-a_6} < 1$. Conseqquently, for all $n \in N$, we obtain:

$$
d_n \leq \lambda d_{n-1} + \lambda^2 d_{n-2} \leq \cdots \leq \lambda^n d_0. \tag{10}
$$

$d_0 = 0$ implies that $p(x_0,x_1)+p(y_0,y_1)=0$. Hence, from Remark 1.1, we obtain $x_0 = x_1 = F(x_0,y_0)$ and $y_0 = y_1 = F(y_0,x_0)$, meaning that (x_0,y_0) is a coupled fixed point of F. If $d_0 > 0$, for all $n \geq m$, we obtain,in view of (p4)

$$
\begin{aligned}
p(x_n,x_m) \quad \leq \quad & p(x_n,x_{n-1})+p(x_{n-1},x_{n-2})-p(x_{n-1},x_{n-1}) \\
& +p(x_{n-2},x_{n-3})+p(x_{n-3},x_{n-4})-p(x_{n-3},x_{n-3}) \\
& +\cdots+p(x_{m+2},x_{m+1})+p(x_{m+1},x_m)-p(x_{m+1},x_{m+1}) \\
\leq \quad & p(x_n,x_{n-1})+p(x_{n-1},x_{n-2})+\cdots+p(x_{m+1},x_m).
\end{aligned} \tag{11}
$$

Similarly, we obtain

$$
p(y_n,y_m) \quad \leq \quad p(y_n,y_{n-1})+p(y_{n-1},y_{n-2})+\cdots+p(y_{m+1},y_m). \tag{12}
$$

Hence,

$$
\begin{aligned}
p(x_n,x_m)+p(y_n,y_m) \quad \leq \quad & d_{n-1}+d_{n-2}+\cdots+d_m \\
\leq \quad & (\lambda^{n-2}+\lambda^{n-2}+\cdots+\lambda^m)d_0 \leq \frac{\lambda^m}{1-\lambda}d_0.
\end{aligned} \tag{13}
$$

Using the definition of p^s, we obtain $p^s(x,y) \leq 2p(x,y)$, hence for each $n \geq m$

$$
p^s(x_n,x_m)+p^s(y_n,y_m) \leq 2p(x_n,x_m)+2p(y_n,y_m) \leq 2\frac{\lambda^m}{1-\lambda}d_0, \tag{14}
$$

this implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,p^s), since $\lambda = \frac{a_1+a_2+a_3+a_6}{1-a_4-a_5-a_6} < 1$. But the PMS (X,p) is complete, and by Lemma 1.1, the metric space (X,p^s) is complete, so there exists $u^*, v^* \in X$ such that

$$
\lim_{n\to+\infty} p^s(x_n,u^*) = \lim_{n\to+\infty} p^s(y_n,v^*) = 0, \tag{15}
$$

using Lemma 1.1, we obtain:

$$
p(u^*,u^*) = \lim_{n\to+\infty} p(x_n,u^*) = \lim_{n\to+\infty} p(x_n,x_n), \tag{16}
$$

$$
p(v^*,v^*) = \lim_{n\to+\infty} p(y_n,v^*) = \lim_{n\to+\infty} p(y_n,y_n). \tag{17}
$$

Using condition (p2) and (10), we obtain:

$$
p(x_n,x_n) \leq p(x_n,x_{n+1}) \leq d_n \leq \lambda^n d_0. \tag{18}
$$

But $\lambda \in [0,1)$, by letting $n \to +\infty$, we obtain $\lim_{n\to+\infty} p(x_n,x_n) = 0$. Hence,

$$
p(u^*,u^*) = \lim_{n\to+\infty} p(x_n,u^*) = \lim_{n\to+\infty} p(x_n,x_n) = 0. \tag{19}
$$

Similarly, we obtain:

$$
p(v^*,v^*) = \lim_{n\to+\infty} p(y_n,v^*) = \lim_{n\to+\infty} p(y_n,y_n) = 0. \tag{20}
$$

Consequently, by (p4) and (4),

$$
\begin{aligned}
p(F(u^*,v^*),u^*) \quad \leq \quad & p(F(u^*,v^*),x_{n+1})+p(x_{n+1},u^*)-p(x_{n+1},x_{n+1}) \\
\leq \quad & p(F(u^*,v^*),F(x_n,y_n))+p(x_{n+1},u^*) \\
\leq \quad & a_1p(u^*,x_n)+a_2p(v^*,y_n)+a_3p(F(u^*,v^*),u^*) \\
& +a_4p(F(x_n,y_n),x_n)+a_5p(F(u^*,v^*),x_n) \\
& +a_6p(F(x_n,y_n),u^*)+p(x_{n+1},u^*) \\
= \quad & a_1p(u^*,x_n)+a_2p(v^*,y_n)+a_3p(F(u^*,v^*),u^*) \\
& +a_4p(x_{n+1},x_n)+a_5p(F(u^*,v^*),x_n) \\
& +a_6p(x_{n+1},u^*)+p(x_{n+1},u^*) \\
\leq \quad & a_1p(u^*,x_n)+a_2p(v^*,y_n)+a_3p(F(u^*,v^*),u^*) \\
& +a_4p(x_{n+1},x_n)+a_4p(u^*,x_n)+a_5p(F(u^*,v^*),u^*) \\
& +a_5p(u^*,x_n)+a_6p(x_{n+1},u^*)+p(x_{n+1},u^*).
\end{aligned} \tag{21}
$$

Now letting $n \to +\infty$, and using (16)-(17), we have:

$$\begin{aligned} p(F(u^*,v^*),u^*) &\le a_3 p(F(u^*,v^*),u^*) + a_5 p(F(u^*,v^*),u^*) \\ &\quad +(a_3+a_5)p(F(u^*,v^*),u^*). \end{aligned} \tag{22}$$

From (22), suppose that $p(F(u^*,v^*),u^*) \neq 0$, so that we can conclude that $1 \le (a_3+a_5)$ which is a contradiction. Hence $p(F(u^*,v^*),u^*) = 0$, i.e $F(v^*,u^*) = v^*$. Hence (u^*,v^*) is a coupled fixed point of F. Next, we prove the uniqueness of the coupled fixed point of F. Suppose that $(u^{'},v^{'})$ is another coupled fixed point of F, then, by using (4),

$$\begin{aligned} p(u^{'},u^*) &= p(F(u^{'},v^{'}),F(u^*,v^*)) \\ &\le a_1 p(u^{'},u^*) + a_2 p(v^{'},v^*) + a_3 p(F(u^{'},v^{'}),u^{'}) \\ &\quad +a_4 p(F(u^*,v^*),u^*) + a_5 p(F(u^{'},v^{'}),u^*) \\ &\quad +a_6 p(F(u^*,v^*),u^{'}) \\ &= a_1 p(u^{'},u^*) + a_2 p(v^{'},v^*) + a_3 p(u^{'},u^{'}) + a_4 p(u^*,u^*) + \\ &\quad a_5 p(u^{'},u^*) + a_6 p(u^*,u^{'}) \\ &\le a_1 p(u^{'},u^*) + a_2 p(v^{'},v^*) + a_3 p(u^{'},u^*) + a_4 p(u^{'},u^*) \\ &\quad +a_5 p(u^{'},u^*) + a_6 p(u^*,u^{'}) \end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned} p(v^{'},v^*) &\le a_1 p(v^{'},v^*) + a_2 p(u^{'},u^*) + a_3 p(v^{'},v^*) \\ &\quad +a_4 p(v^{'},v^*) + a_5 p(v^{'},v^*) + a_6 p(v^*,v^{'}). \end{aligned} \tag{24}$$

Hence,

$$p(u^{'},u^*) + p(v^{'},v^*) \le (a_1+a_2+a_3+a_4+a_5+a_6)[p(u^{'},u^*) + p(v^{'},v^*)] \tag{25}$$

But $a_1+a_2+a_3+a_4+a_5+2a_6 < 1$ implies that $a_1+a_2+a_3+a_4+a_5+a_6 < 1$. This implies that $p(u^{'},u^*) + p(v^{'},v^*) = 0$, hence $u^* = u^{'}$ and $v^* = v^{'}$. Hence, F has a unique coupled fixed point. ■

Theorem 2.1 lead to the following Corollary:

Corollary 2.1. Let (X,p) be a complete partial metric space. Suppose that the mapping $F: X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$

$$\begin{aligned} p(F(x,y),F(u,v)) &\le \frac{a_1}{6}[p(x,u) + p(y,v) + p(F(x,y),x) + p(F(u,v),u) \\ &\quad +p(F(x,y),u) + p(F(u,v),x)] \end{aligned} \tag{26}$$

where $0 \le a_1 < 1$. Then, F has a unique coupled fixed point.

Example 2.1. Let $X = [0,+\infty)$ endowed with the usual partial metric ρ defined by $p: X \times X \to [0,+\infty)$ with $p(x,y) = \max\{x,y\}$. The partial metric space (X,p) is complete because (X,p^s) is complete. Indeed, for any $x,y \in X$,

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\max\{x,y\} - (x+y) = |x-y|, \tag{27}$$

Thus, (X,p^s) is the Euclidean metric space which is complete. Consider the mapping $F: X \times X \to X$ defined by $F(x,y) = \frac{x+y}{12}$. For any $x,y,u,v \in X$, we have

$$\begin{aligned} p(F(x,y),F(u,v)) &= \frac{1}{12}\max\{x+y, u+v, F(x,y)+F(u,v), x+u, \\ &\quad F(x,y)+F(u,v), u+x\} \\ &\le \frac{1}{12}[\max\{x,u\} + \max\{y,v\} + \max\{F(x,y),x\} \\ &\quad +\max\{F(u,v),u\} + \max\{F(x,y),u\} \\ &\quad +\max\{F(u,v),x\}] \\ &= \frac{1}{12}[p(x,u) + p(y,v) + p(F(x,y),x) \\ &\quad +p(F(u,v),u) + p(F(x,y),u) + p(F(u,v),x)]. \end{aligned} \tag{28}$$

Observe that (28) is the contractive condition (26) with $a_1 = \frac{1}{2}$. Hence by Corollary 2.1, F has a unique coupled fixed point, which is (0,0). Observe that if the mapping $F: X \times X \to X$ is given by $F(x,y) = \frac{(x+y)}{6}$, then F satisfies the contractive condition (26) for $a_1 = 1$, i.e

$$\begin{aligned} p\big(F(x,y),F(u,v)\big) &= \frac{1}{6}\max\{x+y, u+v, F(x,y)+F(u,v), x+u, F(x,y)+F(u,v), u+x\} \\ &\le \frac{1}{6}[\max\{x,u\} + \max\{y,v\} + \max\{F(x,y),x\} + \max\{F(u,v),u\} + \max\{F(x,y),u\} + \max\{F(u,v),x\}] \\ &= \frac{1}{6}[p(x,u) + p(y,v) + p(F(x,y),x) + p(F(u,v),u) + p(F(x,y),u) + p(F(u,v),x)] \end{aligned}$$

However, $(0,0)$ and $(1,1)$ are both coupled fixed points of F under this condition that $a_1 = 1$. This shows that the condition $a_1 < 1$ is essentialin Corollary 2.1 and Theorem 2.1 to obtain a unique coupled fixed point of F.

Remark 2.1. Theorem 2.1 is a unification, an extension and a generalization of Theorem 2.1 [3], Theorem 2.4 [3] and Theorem 2.5 [3]. If $a_3 = a_4 = a_5 = a_6 = 0$, then we obtain the results of Theorem 2.1 [3]. If $a_1 = a_2 = a_5 = a_6 = 0$, then we obtain Theorem 2.4 [3]. If $a_1 = a_2 = a_3 = a_4 = 0$, then we obtain Theorem 2.5 [3]. Similarly, Corollary 2.1 extends, unifies and generalizes Corollary 2.2 [3], Corollary 2.6 [3] and Corollary 2.7 [3].

Theorem 2.2. Let (X,p) be a complete partial metric space. Suppose that the mapping $F: X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$

$$p\big(F(x,y), F(u,v)\big) \leq k\ell_{x,y,u,v} \tag{29}$$

where

$$\ell_{x,y,u,v} \quad \in \quad \{p(x,u), p(y,v), p(F(x,y),x), p(F(u,v),u), \frac{p(F(x,y),u) + p(F(u,v),x)}{2}\} \tag{30}$$

and $k \in [0,1)$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Continuing this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then by (29) and (p2) we obtain:

$$p(x_n, x_{n+1}) = p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \leq k\ell_{x,y,u,v}$$

where

$$\begin{aligned}
\ell_{x,y,u,v} \quad &\in \quad \{p(x_{n-1}, x_n), p(y_{n-1}, y_n), p(F(x_{n-1}, y_{n-1}), x_{n-1}),\\
&\qquad p(F(x_n, y_n), x_n), \frac{p(F(x_{n-1}, y_{n-1}), x_n) + p(F(x_n, y_n), x_{n-1})}{2}\}\\
&= \quad \{p(x_{n-1}, x_n), p(y_{n-1}, y_n), p(x_n, x_{n-1}), p(x_{n+1}, x_n),\\
&\qquad \frac{p(x_n, x_n) + p(x_{n+1}, x_{n-1})}{2}\}\\
&\leq \quad \{p(x_{n-1}, x_n), p(y_{n-1}, y_n), p(x_n, x_{n-1}), p(x_{n+1}, x_n),\\
&\qquad \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n-1})}{2}\}\\
&\leq \quad \{p(x_{n-1}, x_n), p(y_{n-1}, y_n), p(x_n, x_{n-1}), p(x_{n+1}, x_n),\\
&\qquad \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_n) + p(x_n, x_{n-1})}{2}\}
\end{aligned} \tag{31}$$

Similarly,

$$p(y_n, y_{n+1}) = p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \leq k\ell_{x,y,u,v} \tag{32}$$

where

$$\begin{aligned}
\ell x, y, u, v \quad &\in \quad \{p(y_{n-1}, y_n), p(x_{n-1}, x_n), p(y_n, y_{n-1}), p(y_{n+1}, y_n),\\
&\qquad \frac{p(y_n, y_{n+1}) + p(y_{n+1}, y_n) + p(y_n, y_{n-1})}{2}\}
\end{aligned} \tag{33}$$

Set $d_n = p(x_n, x_{n+1}) + p(y_n, y_{n+1})$.

Case 1. If $\ell_{x,y,u,v} = p(x_{n-1}, x_n) + p(y_{n-1}, y_n)$, then $d_n \leq p(x_{n-1}, x_n) + p(y_{n-1}, y_n) = d_{n-1} \leq kd_{n-1}$. Hence, (29) is satisfied.

Case 2. If $\ell_{x,y,u,v} = p(x_{n+1}, x_n) + p(y_{n+1}, y_n)$, then $d_n = p(x_{n+1}, x_n) + p(y_{n+1}, y_n) \leq kd_n$. Hence, (29) is satisfied.

Case 3. If $\ell_{x,y,u,v} = \frac{p(x_n,x_{n+1})+p(x_{n+1},x_n)+p(x_n,x_{n-1})}{2} + \frac{p(y_n,y_{n+1})+p(y_{n+1},y_n)+p(y_n,y_{n-1})}{2}$ then

$$\begin{aligned}
d_n \quad &\leq \quad \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_n) + p(x_n, x_{n-1})}{2} + \frac{p(y_n, y_{n+1}) + p(y_{n+1}, y_n) + p(y_n, y_{n-1})}{2}\\
&= \quad \frac{2p(x_{n+1}, x_n) + p(x_n, x_{n-1})}{2} + \frac{2p(y_{n+1}, y_n) + p(y_n, y_{n-1})}{2}\\
&\leq \quad p(x_n, x_{n+1}) + \frac{p(x_n, x_{n-1})}{2} + p(y_n, y_{n+1}) + \frac{p(y_n, y_{n-1})}{2}.
\end{aligned} \tag{34}$$

Hence, from (34) we obtain $0 \leq \frac{p(x_n,x_{n-1})}{2} + \frac{p(y_n,y_{n-1})}{2}$, this implies that $0 \leq p(x_n, x_{n-1}) + p(y_n, y_{n-1}) = d_{n-1}$. Hence (29) is satisfied in all cases.

Conseqquently, for all $n \in N$, we obtain:

$$d_n \leq kd_{n-1} + k^2 d_{n-2} \leq \cdots \leq k^n d_0. \tag{35}$$

$d_0 = 0$ implies that $p(x_0, x_1) + p(y_0, y_1) = 0$. Hence, from Remark 1.1, we obtain $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$, meaning that (x_0, y_0) is a coupled fixed point of F. If $d_0 > 0$, for all $n \geq m$, we obtain,in view of (p4)

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) - p(x_{n-1}, x_{n-1}) \\ &\quad + p(x_{n-2}, x_{n-3}) + p(x_{n-3}, x_{n-4}) - p(x_{n-3}, x_{n-3}) \\ &\quad + \cdots + p(x_{m+2}, x_{m+1}) + p(x_{m+1}, x_m) - p(x_{m+1}, x_{m+1}) \\ &\leq p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots + p(x_{m+1}, x_m). \end{aligned} \tag{36}$$

Similarly, we obtain:

$$p(y_n, y_m) \leq p(y_n, y_{n-1}) + p(y_{n-1}, y_{n-2}) + \cdots + p(y_{m+1}, y_m). \tag{37}$$

Hence,

$$\begin{aligned} p(x_n, x_m) + p(y_n, y_m) &\leq d_{n-1} + d_{n-2} + \cdots + d_m \\ &\leq (k^{n-2} + k^{n-2} + \cdots + k^m) d_0 \\ &\leq \frac{k^m}{1-k} d_0. \end{aligned} \tag{38}$$

Using the definition of p^s, we obtain $p^s(x, y) \leq 2p(x, y)$, hence for each $n \geq m$

$$p^s(x_n, x_m) + p^s(y_n, y_m) \leq 2p(x_n, x_m) + 2p(y_n, y_m) \leq 2\tfrac{k^m}{1-k} d_0, \tag{39}$$

this implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, p^s), since $k \in [0,1)$.

But the PMS (X, p) is complete, and by Lemma 1.1, the metric space (X, p^s) is complete, so there exists $u^*, v^* \in X$ such that

$$\lim_{n\to+\infty} p^s(x_n, u^*) = \lim_{n\to+\infty} p^s(y_n, v^*) = 0, \tag{40}$$

using Lemma 1.1, we obtain:

$$p(u^*, u^*) = \lim_{n\to+\infty} p(x_n, u^*) = \lim_{n\to+\infty} p(x_n, x_n), \tag{41}$$

$$p(v^*, v^*) = \lim_{n\to+\infty} p(y_n, v^*) = \lim_{n\to+\infty} p(y_n, y_n). \tag{42}$$

Using condition (p2) and (35), we obtain:

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \leq d_n \leq \lambda^n d_0. \tag{43}$$

But $\lambda \in [0,1)$, by letting $n \to +\infty$, we obtain $\lim_{n\to+\infty} p(x_n, x_n) = 0$. Hence,

$$p(u^*, u^*) = \lim_{n\to+\infty} p(x_n, u^*) = \lim_{n\to+\infty} p(x_n, x_n) = 0. \tag{44}$$

Similarly, we obtain:

$$p(v^*, v^*) = \lim_{n\to+\infty} p(y_n, v^*) = \lim_{n\to+\infty} p(y_n, y_n) = 0. \tag{45}$$

Hence, by using (29) and (p4), we obtain:

$$\begin{aligned} p(F(u^*, v^*), u^*) &\leq p(F(u^*, v^*), x_{n+1}) + p(x_{n+1}, u^*) - p(x_{n+1}, x_{n+1}) \\ &\leq p(F(u^*, v^*), F(x_n, y_n)) + p(x_{n+1}, u^*) \\ &\leq k\ell_{x_n, y_n, u^*, v^*} + p(x_{n+1}, u^*), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \ell_{x_n, y_n, u^*, v^*} &\in \{p(u^*, x_n), p(v^*, y_n), p(F(u^*, v^*), u^*), \\ &\quad p(F(x_n, y_n), x_n), \frac{p(F(u^*, v^*), x_n) + p(F(x_n, y_n), u^*)}{2}\} \\ &= \{p(u^*, x_n), p(v^*, y_n), p(F(u^*, v^*), u^*), \\ &\quad p(x_{n+1}, x_n), \frac{p(F(u^*, v^*), x_n) + p(x_{n+1}, u^*)}{2}\}. \end{aligned} \tag{47}$$

We now consider the following cases:

Case 1^0. If $\ell_{x_n, y_n, u^*, v^*} = p(u^*, x_n)$, then from (46) we obtain:

$$p(F(u^*, v^*), u^*) \leq kp(u^*, x_n) + p(x_{n+1}, u^*). \tag{48}$$

By letting $n \to +\infty$ and using (44)-(45), we obtain $p(F(u^*, v^*), u^*) = 0$.

Case 2^0. If $\ell_{x_n, y_n, u^*, v^*} = p(v^*, y_n)$, then from (46), we obtain:

$$p(F(u^*, v^*), u^*) \leq kp(v^*, y_n) + p(x_{n+1}, u^*). \tag{49}$$

By letting $n \to +\infty$ and using (44)-(45), we obtain $p(F(u^*, v^*), u^*) = 0$.

Case 3^0. If $\ell_{x_n, y_n, u^*, v^*} = p(F(u^*, v^*), u^*)$, then from (46), we obtain:

$$p(F(u^*, v^*), u^*) \leq kp(F(u^*, v^*), u^*) + p(x_{n+1}, u^*). \quad (50)$$

By letting $n \to +\infty$ and using (44), we obtain:

$$p(F(u^*, v^*), u^*) \leq kp(F(u^*, v^*), u^*). \quad (51)$$

From (51), we can obtain a contradiction if we assume that $p(F(u^*, v^*), u^*) \neq 0$, this implies that $1 \leq k$ which ultimately gives a contradiction, hence $p(F(u^*, v^*), u^*) = 0$.

Case 4^0. If $\ell_{x_n, y_n, u^*, v^*} = p(x_{n+1}, x_n)$, then from (46), we obtain:

$$\begin{aligned} p(F(u^*, v^*), u^*) &\leq kp(x_{n+1}, x_n) + p(x_{n+1}, u^*) \\ &\leq kp(x_{n+1}, u^*) + kp(u^*, x_n) + p(x_{n+1}, u^*). \end{aligned} \quad (52)$$

By letting $n \to +\infty$ and using (44)-(45), we obtain $p(F(u^*, v^*), u^*) = 0$.

Case 5^0. If $\ell_{x_n, y_n, u^*, v^*} = \frac{p(F(u^*, v^*), x_n) + p(x_{n+1}, u^*)}{2}$, then from (46), we have:

$$\begin{aligned} p(F(u^*, v^*), u^*) &\leq k[\frac{p(F(u^*, v^*), x_n) + p(x_{n+1}, u^*)}{2}] + p(x_{n+1}, u^*) \\ &\leq k(\frac{p(F(u^*, v^*), u^*)}{2}) + k(\frac{p(u^*, x_n)}{2}) + k(\frac{p(x_{n+1}, u^*)}{2}) \\ &\quad + p(x_{n+1}, u^*). \end{aligned} \quad (53)$$

By letting $n \to +\infty$ and using (44)-(45), we obtain

$$p(F(u^*, v^*), u^*) \leq k(\frac{p(F(u^*, v^*), u^*)}{2}). \quad (54)$$

Thus, by Case 3^0, we have $p(F(u^*, v^*), u^*) = 0$ since $k < 1$ implies that $\frac{k}{2} < 1$.

Hence in all cases, we have $p(F(u^*, v^*), u^*) = 0$, that is $F(u^*, v^*) = u^*$.

Similarly, we obtain $F(v^*, u^*) = v^*$, this means that (u^*, v^*) is a coupled fixed point of F. Next, we prove that the coupled fixed point of F is unique. Suppose that (u', v') is another coupled fixed point of F, then, in view of (29), we have:

$$\begin{aligned} p(u', u^*) &= p(F(u', v'), F(u^*, v^*)) \\ &\leq k\ell_{u', u^*}. \end{aligned} \quad (55)$$

where,

$$\begin{aligned} \ell_{u', u^*} \in \; &\{p(u', u^*), p(v', v^*), p(F(u', v'), u'), p(F(u^*, v^*), u^*), \\ &\frac{p(F(u', v'), u^*) + p(F(u^*, v^*), u')}{2}\}. \end{aligned} \quad (56)$$

Similarly,

$$\begin{aligned} p(v', v^*) &= p(F(v', u'), F(v^*, u^*)) \\ &\leq k\ell_{v', v^*} \end{aligned} \quad (57)$$

where,

$$\begin{aligned} \ell_{v', v^*} \in \; &\{p(v', v^*), p(u', u^*), p(F(v', u'), v'), p(F(v^*, u^*), v^*), \\ &\frac{p(F(v', u'), v^*) + p(F(v^*, u^*), v')}{2}\}. \end{aligned} \quad (58)$$

Adding (55) and (57), we obtain $p(u', u^*) + p(v', v^*)$.

We now consider the following cases:

Case 1^1. If $\ell_{u', u^*} = p(u', u^*), \ell_{v', v^*} = p(v', v^*)$ then from (55) and (57), we obtain:

$$p(u', u^*) + p(v', v^*) \leq k[p(u', u^*) + p(v', v^*)]. \quad (59)$$

But $k \in [0,1)$, implies that $p(u', u^*) + p(v', v^*) = 0$. Hence, $u^* = u'$ and $v^* = v'$.

Case 2^1. If $\ell_{u', u^*} = p(v', v^*), \ell_{v', v^*} = p(u', u^*)$, then from (55) and (57), we obtain:

$$p(u', u^*) + p(v', v^*) \leq k[p(v', v^*) + p(u', u^*)]. \quad (60)$$

But $k \in [0,1)$, implies that $p(u', u^*) + p(v', v^*) = 0$. Hence, $u^* = u'$ and $v^* = v'$.

Case 3^1. If $\ell_{u', u^*} = p(F(u', v'), u'), \ell_{v', v^*} = p(F(v', u'), v')$, then we have

$$\begin{aligned} p(u', u^*) + p(v', v^*) &\leq k[p(F(u', v'), u') + p(F(v', u'), v')] \\ &= k[p(u', u') + p(v', v')] \\ &\leq k[p(u', u^*) + p(v', v^*)]. \end{aligned} \quad (61)$$

But $k \in [0,1)$, implies that $p(u^{'}, u^{*}) + p(v^{'}, v^{*}) = 0$. Hence, $u^{*} = u^{'}$ and $v^{*} = v^{'}$.

$Case \quad 4^{1}$. If $\ell_{u^{'},u^{*}} = p(F(u^{*}, v^{*}), u^{*}), \ell_{v^{'},v^{*}} = p(F(v^{*}, u^{*}), v^{*})$, then we have

$$\begin{aligned} p(u^{'}, u^{*}) + p(v^{'}, v^{*}) &\leq k[p(F(u^{*}, v^{*}), u^{*}) + p(F(v^{*}, u^{*}), v^{*})] \\ &= k[p(u^{*}, u^{*}) + p(v^{*}, v^{*})] \\ &\leq k[p(u^{'}, u^{*}) + p(v^{'}, v^{*})]. \end{aligned} \tag{62}$$

But $k \in [0,1)$, implies that $p(u^{'}, u^{*}) + p(v^{'}, v^{*}) = 0$. Hence, $u^{*} = u^{'}$ and $v^{*} = v^{'}$.

$Case \quad 5^{1}$. If

$\ell_{u^{'},u^{*}} = \frac{p(F(u^{'},v^{'}),u^{*}) + p(F(u^{*},v^{*}),u^{'})}{2}, \ell_{v^{'},v^{*}} = \frac{p(F(v^{'},u^{'}),v^{*}) + p(F(v^{*},u^{*}),v^{'})}{2}$, then we obtain:

$$\begin{aligned} p(u^{'}, u^{*}) + p(v^{'}, v^{*}) &\leq k[\frac{p(F(u^{'}, v^{'}), u^{*}) + p(F(u^{*}, v^{*}), u^{'})}{2} \\ &\quad + \frac{p(F(v^{'}, u^{'}), v^{*}) + p(F(v^{*}, u^{*}), v^{'})}{2}] \\ &= k[\frac{p(u^{'}, u^{*}) + p(u^{*}, u^{'})}{2} + \frac{p(v^{'}, v^{*}) + p(v^{*}, v^{'})}{2}] \\ &= k[p(u^{'}, u^{*}) + p(v^{'}, v^{*})] \end{aligned} \tag{63}$$

But $k \in [0,1)$, implies that $p(u^{'}, u^{*}) + p(v^{'}, v^{*}) = 0$. Hence, $u^{*} = u^{'}$ and $v^{*} = v^{'}$.

Hence, in all cases, we have established that $u^{*} = u^{'}$ and $v^{*} = v^{'}$. Hence (u^{*}, v^{*}) is a unique coupled fixed point of F.

Remark 2.2. Theorem 2.2 is a unification, an extension and a generalization of the results of [12]. Since [12, Theorem 2.2], [12, Corollary 2.3], [12, Theorem 2.6], [12, Corollary 2.7] and [12, Corollary 2.8] are all special cases of Theorem 2.2.

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References

[1] *Abdeljawad T., Erdal Karapinar and Tas K., Existence and uniqueness of a common fixed point on partial metric spaces, Applied Mathematics Letters, 24(2011), pp. 1900-1904.*

[2] *Altun I., Sola F. and Simsek H., it Generalized contractions on partial metric spaces, Topology and its Applications, 157 (2010), pp. 2778-2785.*

[3] *Aydi H., Some coupled fixed point results on partial metric spaces, International Journal of Mathematics and Mathematical Sciences, vol. 2011, Article ID 647091, 11 pages.*

[4] *Bhaskar T. G. and Lakshmikantham V., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Methods & Applications, 70(7)(2006), pp. 1379- 1393.*

[5] *Ciric L. and Lakshmikantham V., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Methods & Applications, 70(12) (2009), pp. 4341-4349.*

[6] *Ilic D., Pavlovic V. and Rakocevic V., Some new extensions of Banach's contraction principle to partial metric space, Applied Mathematics Letters, 24(2011), pp. 1326-1330.*

[7] *Matthews S. G., Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in Annals of the New York Academy of Sciences, 728 (1994), pp. 183-197.*

[8] *Olaleru J. O., Okeke G. A. and Akewe H., Coupled fixed point theorems for generalized ϕ-mappings satisfying contractive condition of integral type on cone metric spaces, International Journal of Mathematical Modelling & Computations 02(02)(2012),pp. 87-98.*

[9] *Rarison A. F., in: Hans-Peter Kunzi A. (Ed.) Partial Metrics, African Institute for Mathematical Sciences, (2007), Supervised.*

[10] *Romaguera S., A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory and Applications, 2010, Article ID 493298, 6 pages, 2010.*

[11] *Romaguera S. and Schellekens M., Partial metric monoids and semivaluation spaces, Topology and its Applications, 153(5-6)(2005), pp. 948-962.*

[12] *Sabetghadam F., Masiha H. P. and Sanatpour A. H., Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory and Applications, Article ID 125426, 8 pages, 2009.*