Local Fractional Operator for the Solution of Quadratic Riccati Differential Equation with Constant Coefficients

Sunday O. EDEKI †, *Member, IAENG,* Grace O. AKINLABI, Kennenth ONYENIKE

Abstract— **In this paper, we consider approximate solutions of fractional Riccati differential equations via the application of local fractional operator in the sense of Caputo derivative. The proposed semi-analytical technique is built on the basis of the standard Differential Transform Method (DTM). Some illustrative examples are given to demonstrate the effectiveness and robustness of the proposed technique; the approximate solutions are provided in the form of convergent series. This shows that the solution technique is very efficient, and reliable; as it does not require much computational work, even without given up accuracy.**

Index Terms— **Fractional differential equations; Modified DTM; Riccati Differential Equation**

I. Introduction

THE idea of Riccati Differential Equation (RDE) was introduced by the Italian: Count Jacopo Francesco Riccati (1676-1754), the detail of the basic theories associated with the RDEs are contained in Reid [1]. The applications of this form of differential equations are not limited to areas like random processes, diffusion problems, optimal control, stochastic realization theory, network analysis, financial mathematics, and robust stabilization [2,3].

In general, the RDE of integer order is expressed in the form of:

$$\frac{dy}{dt} = \sum_{i=0}^{2} p_i(t) y^i, \ t \geq 0, \ y(0) = y_0 \qquad (1.1)$$

where $p_i(t), \ i = 0,1,2.$ are coefficient functions which

may be constants or variables; hence, the notion of Riccati differential equation with constant or variable coefficients.

Equation (1.1) is a nonlinear differential equation whose approximate/analytical solutions and the likes can be

obtained using some semi-analytical methods, say the decomposition method (ADM), Homotopy analysis method (HAM), Differential transformation method (DTM), Projected/Modified Differential transform method (MDTM), Forward-Euler method (FEM), Runge-Kutta method (R-KM), Perturbation Iteration Transform Method and so on [4-11].

Recently, Biazar and Eslami [12] proposed DTM for solving the quadratic RDE of the form (1.1) but with constant coefficients while Mukherjee and Roy [13] implemented the DTM for the solution of some RDEs with variable terms as coefficients.

In this work, we shall be considering the extension of (1.1) to fractional differential case, therefore, the fractional RDE (FRDE) of the form:

$$\frac{d^\alpha y}{dt^\alpha} = \sum_{i=0}^{2} p_i(t) y^i, \ t \geq 0, \ \alpha \in (0,1] \qquad (1.2)$$

where $y(0) = y_0$ is an initial condition, and $p_i(t), \ i = 0,1,2.$ are constant functions. Thus, a FRDE with constant coefficients.

Differential equations of fractional type (FDEs) act as generalizations of the classical differential equations of integer order [14].

Many researchers have considered obtaining approximate solutions of FDEs of the form in (1.2) and the likes via the application of semi-analytical methods [15-21].

In this present work, we will be considering the application of a local fractional differential operator (LFDO) based on MDTM for approximate solution of the FRDE in (1.2). This method involves less computational work, and requires less computational time.

II. Preliminaries and Notations on Fractional Calculus [14, 22, 23]

Here, a brief introduction of fractional calculus will be given as follows.

In fractional calculus, the power of the differential operator is considered a real or complex number. Hence, the following definitions:

Definition 1: Fractional derivative in gamma sense

Suppose $D = \frac{d(\cdot)}{dx}$ and J are differential and integration operators respectively, then:

$$D^\alpha h(x) = \frac{d^\alpha h(x)}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}. \qquad (2.1)$$

We referred to (2.1) as a fractional derivative of $h(x)$, of order α, if $\alpha \in \mathbb{R}$.

Definition 2: Suppose $h(x)$ is defined for $x > 0$, then:

$$(Jh)(x) = \int_0^x h(s)ds \qquad (2.2)$$

and as such, an arbitrary extension of (2.2) yields:

$$(J^\alpha h)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}h(s)ds, \; \alpha > 0, \; t > 0. \qquad (2.3)$$

Equation (2.3) is regarded as Riemann-Liouville (R-L) fractional integration of order α.

Definition 3: R-L fractional derivative of ℓ is:

$$D^\alpha \ell \qquad \frac{{}^{J^\phi}\left(J^{\phi-\alpha}\ell\right)}{dx^\phi}. \qquad (2.4)$$

Definition 4: Caputo fractional derivative of ℓ is:

$$D^\alpha \ell \qquad {}^{J^{\phi-\alpha}\left(J^\phi \ell\right)} \qquad \mathbb{N}. \qquad (2.5)$$

Note: The link between the R-L operator and the Caputo fractional differential operator is:

$$(J^\alpha D_t^\alpha)\ell \qquad \ell \quad \ell \quad \sum_{k=0}^{n-1}\ell \qquad {}^{.k} \quad N:$$

$n-1 < \alpha < n, \; n \in \mathbb{N}$. $\qquad (2.6)$

As such,

$$\ell \qquad \ell \quad \sum_{k=0}^{n-1}\ell \qquad {}^{.k} \quad N: . \qquad (2.7)$$

Definition 5: The Mittag-Leffler (M-L) Function

The M-L function, $E_\alpha(z)$ is defined and denoted by the series representation as:

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha \geq 0, \; z \in \mathbb{C}. \qquad (2.8)$$

Remark: For $\alpha = 1$, $E_\alpha(z)$ in (2.8) becomes:

$$E_{\alpha=1}(z) = e^z. \qquad (2.9)$$

III. ANALYSIS OF THE DTM

The semi-analytical method (DTM) as noted by many researchers in literature, has proven to be simple and easier in the sense of application for both linear and nonlinear differential models because the DTM converts the problems under consideration to their equivalent forms in algebraic recursive relations, but this is not so when other semi-analytical techniques, say, ADM, VIM, HAM and so on are used) [24]. The DTM has been modified to handle models of nonlinear types and the likes [25-28].

A. The overview of the DTM

Let $v(x)$ be an analytic function at x_+ in a domain D, thus, the differential transform (DF) of $v(x)$ is defined and denoted by:

$$V(p) = \frac{1}{p!}\left[\frac{d^p v(x)}{dx^p}\right]_{x=x_+}. \qquad (3.1)$$

And as such:

$$v(x) = \sum_{p=0}^{\infty}V(p)(x-x_+)^p \qquad (3.2)$$

Equation (3.2) is the differential inverse transform (DIT) of $V(p)$, where $v(x)$ and $V(p)$ are the original and the transformed functions respectively.

B. The fundamentals of DTM: theorems and properties

Theorem 1: If $v(x) = \alpha v_a(x) \pm \beta v_b(x)$, then

$$V(p) = \alpha V_a(p) \pm \beta V_b(p).$$

Theorem 2 : If $v(x) = \dfrac{\alpha d^\eta v_+(x)}{dx^\eta}$, $\eta \in \mathbb{N}$, then

$$V(p) = \frac{\alpha(p+\eta)!}{p!}V_+(p+\eta).$$

Theorem 3: If $v(x) = \dfrac{\alpha dv_+(x)}{dx}$, then

$$V(p) = \frac{\alpha(p+1)!V_+(p+1)}{p!} = \alpha(p+1)V_+(p+1).$$

Theorem 4: If $v(x) = v_+^2(x)$, then

$$V(p) = \sum_{\eta=0}^{p}V_+(\eta)V_+(p-\eta).$$

Theorem 5: (PDTM of a fractional derivative)
If $f(x) = D_x^\alpha w(x)$, then

$$\Gamma\left(1+\frac{p}{q}\right)F(p) = \Gamma\left(1+\alpha+\frac{p}{q}\right)W(p+\alpha q).$$

Consequently, we have:

$$\Gamma\left(1+\alpha+\frac{p}{q}\right)W(p+\alpha q) = \Gamma\left(1+\frac{p}{q}\right)F(p). \qquad (3.3)$$

Setting $\alpha q = 1$ in (3.3) gives:

$$W(p+1) = \frac{\Gamma(1+\alpha p)}{\Gamma(1+\alpha(1+p))}F(p). \qquad (3.4)$$

As such, for $w(x)$, α-analytic at $x_0 = 0$

$$w(x) = \sum_{\eta=0}^{\infty}W(\eta)x^{\frac{\eta}{q}} = \sum_{\eta=0}^{\infty}W(\eta)x^{\alpha\eta}. \qquad (3.5)$$

C. Analysis of the Fractional DTM

Consider the nonlinear fractional differential equation (NLFDE):

$$D_x^\alpha w(x) + L_{\{x\}}w(x) + N_{\{x\}}w(x) = q_+(x),$$

$$w(x,0) = g_+(x), \; x > 0 \qquad (3.6)$$

where $D_x^\alpha = \dfrac{d^\alpha}{dt^\alpha}$ is the fractional Caputo derivative of $w = w(x)$; whose projected differential transform is $W(p)$ while $L_{\{x\}}$ and $N_{\{x\}}$ are differential operators (with respect to x) of linear and nonlinear type respectively, and $q_+ = q_+(x)$ is the source term.

We rewrite (3.6) as:

$$\begin{cases} D_x^\alpha w(x) = -L_{\{x\}} w(x) - N_{\{x\}} w(x) q_+(x), \\ w(0) = g_+(x), \ \eta - 1 < \alpha < \eta , \ \eta \in \mathbb{N} \end{cases} \quad (3.7)$$

Applying the inverse fractional Caputo derivative, $D_x^{-\alpha}$ to both sides of (3.7) and with regard to (2.6) gives:

$$\begin{cases} w(x) = g(x) + D_x^{-\alpha}\left[-L_{[x]} w(x) - N_{[x]} w(x) + q(x) \right] \\ w(0) = g(x). \end{cases}$$

(3.8)

Thus, when $w(x)$ is expanded in terms of fractional power series, the inverse projected differential transform of $W(p)$ is given as follows:

$$\begin{cases} w(x) = \sum_{\eta=0}^{\infty} W(\eta) x^{\alpha\eta} = w(0) + \sum_{\eta=1}^{\infty} W(\eta) x^{\alpha\eta}, \\ w(x,0) = g_+(x). \end{cases} \quad (3.9)$$

IV. ILLUSTRATIVE EXAMPLES AND APPLICATIONS

In this subsection, the proposed method is applied with some illustrative examples for the solutions of fractional Riccati differential equations (FRDEs).

Problem 4.1: Consider the FRDE in (1.2) with $p_0(t) = 1$, $p_1(t) = 0$, & $p_2(t) = -1$, thus, we have:

$$D_t^\alpha y(t) + y^2(t) - 1 = 0 , \ \& \ y(0) = 0. \quad (4.1)$$

Solution to problem 4.1:
We re-write (4.1) as:

$$D_t^\alpha y(t) = 1 - y^2(t) , \ \& \ y(0) = 0. \quad (4.2)$$

Taking the LFDT of (4.2) gives:

$$\begin{cases} LFDT\left[D_t^\alpha y(t) = 1 - y^2(t) \right], \\ LFDT\left[y(0) = 0 \right]. \end{cases} \quad (4.3)$$

$$\Rightarrow$$

$$\begin{cases} \dfrac{\Gamma(1+\alpha(1+k))}{\Gamma(1+\alpha k)} Y(1+k) = \left(\delta(k) - \sum_{\eta=0}^{k} Y(\eta) Y(k-\eta) \right), \\ Y(0) = 0. \end{cases}$$

(4.4)

Thus, for $k=0$, $k=1$, $k=2$, $k=3$, $k=4$, $k=5 \cdots$, we have respectively $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, \cdots$ as follows:

$$Y(1) = Y_1 = \frac{1}{\Gamma(1+\alpha)} \left\{ 1 - Y^2(0) \right\}, \quad (4.5)$$

$$Y(2) = Y_2 = \frac{-2\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \{ Y_0 Y_1 \}, \quad (4.6)$$

$$Y(3) = Y_3 = \frac{-\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \{ 2Y_0 Y_2 + Y_1 Y_1 \}, \quad (4.7)$$

$$Y(4) = Y_4 = \frac{-2\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \{ Y_0 Y_3 + Y_1 Y_2 \}, \quad (4.8)$$

$$Y(5) = Y_5 = \frac{-\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)} \{ 2Y_0 Y_4 + 2Y_1 Y_3 + Y_2 Y_2 \},$$

(4.9)

$$Y(6) = Y_6 = \frac{-2\Gamma(1+5\alpha)}{\Gamma(1+6\alpha)} \{ Y_0 Y_5 + Y_1 Y_4 + Y_2 Y_3 \}.$$

(4.10)

Hence, using the initial condition, we obtain:

$$Y_0 = Y_2 = Y_4 = Y_6 = Y_8 = \cdots \qquad \mathbb{N},$$

$$Y_1 = \frac{1}{\Gamma(1+\alpha)}, \qquad Y_3 = \frac{-\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{1}{\Gamma(1+\alpha)} \right)^2,$$

$$Y_5 = \frac{2\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)} \left\{ \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{1}{\Gamma(1+\alpha)} \right)^3 \right) \right\}, \cdots$$

$$\therefore \ y(t) = \sum_{h}^{\infty} Y_h t^{\alpha h}$$

$$= \frac{1}{\Gamma(1+\alpha)} t^\alpha - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{1}{\Gamma(1+\alpha)} \right)^2 t^{3\alpha}$$

$$+ \frac{2\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)} \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{1}{\Gamma(1+\alpha)} \right)^3 t^{5\alpha} + \cdots$$

(4.11)

Remark (a): For $\alpha = 1$, we have
$$y(t) = t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \cdots$$ which corresponds to the solution as contained in [4, 13] via a decomposed method.
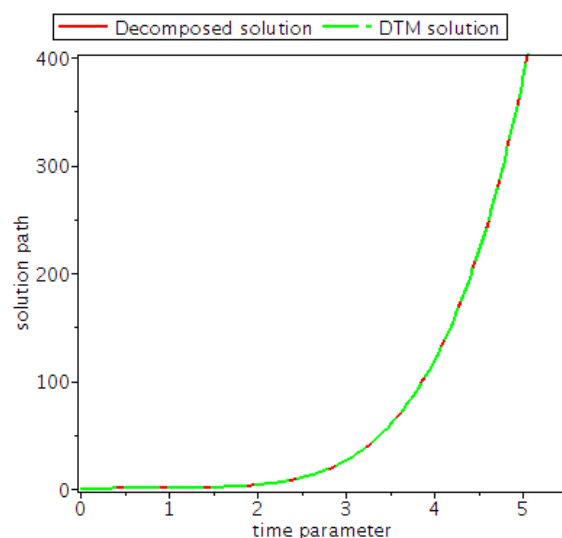

Fig.1: solution of 4.1 via decomposed and DT methods

V. CONCLUDING REMARKS

We have considered in this work, the approximate solutions of fractional Riccati differential equations (FRDEs) via the application of the local fractional operator in the sense of Caputo derivative as a proposed semi-analytical method built on the basis of the classical DTM. To demonstrate the effectiveness and robustness of the present technique, we used some illustrative examples; the solutions are provided in the form of convergent series. This supports the efficiency, and reliability of the solution method; as it does not require much computational work, even without given up accuracy. We remarked that the solutions of RDEs at different (well-defined) values of fractional orders can easily be obtained. Thus, the method is recommended for the solutions linear and nonlinear time-fractional differential equations (TFDEs) with wider applications in other areas of applied sciences.

REFERENCES

[1] W. T. Reid, "Riccati differential equations" *Mathematics in science and engineering*, New York: Academic Press. **86**:1972.

[2] I. Lasiecka and R.Triggiani, "Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory", *Lecture notes in control and information sciences,* Berlin: Springer. **164**, (1991).

[3] F. Dubois and A. Saidi, "Unconditionally stable scheme for Riccati equation", ESAIM Proceedings, **8** (2000):39-52.

[4] A. A. Bahnasawi, M. A. El-Tawil and A. Abdel-Naby, "Solving Riccati differential equation using Adomian's decomposition method", *Appl. Math. Comput.*, **157** (2004): 503-514.

[5] S.O. Edek**i**, O.O. Ugbebor, E.A. Owoloko, "He's Polynomials for Analytical Solutions of the Black-Scholes Pricing Model for Stock Option Valuation", *Proceedings of the World Congress on Engineering,* 2016, Vol II, WCE 2016, June 29 – July 1, London, UK.

[6] G.O. Akinlabi and S. O. Edeki, The solution of initial-value wave-like models via perturbation iteration transform method, *International MultiConference of Engineers and Computer Scientists,* Hong Kong, 15-17 March 2017, (Accepted-2017).

[7] Y. Tan and S. Abbasbandy, "Homotopy analysis method for quadratic Riccati differential equation", *Commun. Nonlin. Sci. Numer. Simul.,* 13 (3), (2008):539-546.

[8] S.O. Edeki, G.O. Akinlabi, A.O. Akeju, "A Handy Approximation Technique for Closed-form and Approximate Solutions of Time-Fractional Heat and Heat-Like Equations with Variable Coefficients", *Proceedings of the World Congress on Engineering*, 2016, Vol II, WCE 2016, June 29 – July 1, London, UK.

[9] S.O. Edeki, G.O. Akinlabi, M.E. Adeosun, "Analytical Solutions of the Navier-Stokes Model by He's Polynomials", *Proceedings of the World Congress on Engineering, 2016,* Vol I, WCE 2016, June 29 – July 1, London, UK.

[10] S. Abbasbandy, "A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials", *J. Comput. Appl. Math.*, **207** (1), (2007):59-63.

[11] B. Batiha, M. S. M. Noorani and I. Hashim. "Application of Variational Iteration Method to a General Riccati Equation", *International Mathematical Forum*, **2** (56), (2007): 2759-2770.

[12] J. Biazar, M. Eslami. "Differential Transform Method for Quadratic Riccati Differential Equation", *.International Journal of Nonlinear Science,* **9** (4), (2010):444-447.

[13] S. Mukherjee and B. Roy, "Solution of Riccati equation with variable co-efficient by Differential transform method*", International journal of nonlinear science*, **14** (2), (2012): 251-256.

[14] I. Podlubny, "Fractional Differential Equations", *Academic Press*, (1999).

[15] A. Elsaid, M. S. Abdel Latif, F. M. Kamal, "Series Solution for Fractional Riccati Differential Equation and tis Convergence", *Journal of Fractional Calculus and Applications,* **6** (2), (2015): 186-196.

[16] S. Momani, N. Shawagfeh, "Decomposition method for solving fractional Riccati differential equations", *Appl. Math. Comput.,* **182** (2006): 1083-1092.

[17] N. Khan, A. Ara, M. Jamil, "An effcient approach for solving the riccati equation with fractional orders", *Comput.Math. Appl.,* **61** (2011) 2683-2689.

[18] Y. Li, "Solving a nonlinear fractional differential equation using Chebyshev wavelets", *Commun. Nonlinear Sci. Numer. Simul.,* **15** (2010) 2284-2292.

[19] Z. Odibat, S. Momani, "Modifed Homotopy perturbation method: application to quadratic Riccati differential equation of fractional order", *Chaos Solitons fractals*, **36** (2008): 167-174.

[20] J. Cang, Y. Tan, H. Xu, S. Liao, "Series solutions of non-linear Riccati differential equations with fractional order", *Chaos Solitons fractals,* **40,** (2009): 1-9.

[21] N. Sweilam, M. Khader, A. Mahdy, "Numerical studies for solving fractional Riccati differential equation", *Appl. Appl. Math*., **7**, (2012): 595-608.

[22] S.O. Edeki, G.O. Akinlabi and S.A. Adeosun, "Analytic and Numerical Solutions of Time-Fractional Linear Schrödinger Equation", *Communications in Mathematics and Applications*, **7** (1), (2016): 1–10.

[23] M. Dalir, "Applications of Fractional Calculus", *Applied Mathematical Sciences,* **4**,(21), (2010): 1021-1032.

[24] J.K.Zhou, Differential Transformation and its Application in Electrical Circuits, Huazhong University Press,Wuhan, China, (1986).

[25] S.O. Edeki, G.O. Akinlabi, S.A. Adeosun, "On a modified transformation method for exact and approximate solutions of linear Schrödinger equations*", AIP Conference proceedings* **1705**, 020048 (2016); doi: 10. 1063/1.4940296.

[26] B. Jang, "Solving linear and nonlinear initial value problems by the projected differential transform method", *Computer Physics Communications* **181** (2010), 848-854.

[27] S.O. Edeki, O.O. Ugbebor, E.A. Owoloko, "Analytical Solutions of the Black–Scholes Pricing Model for European Option Valuation via a Projected Differential Transformation Method" Entropy, (2015), 17 (11), 7510-7521.

[28] G.O. Akinlabi and S. O. Edeki, On Approximate and Closed-form Solution Method for Initial-value Wave-like Models, *International Journal of Pure and Applied Mathematics,* **107** (2), (2016): 449-456.