GENERALISATION OF THE INVERSE EXPONENTIAL DISTRIBUTION: STATISTICAL PROPERTIES AND APPLICATIONS

BY

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Matriculation Number: 13PCD00573

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A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY, COVENANT UNIVERSITY, OTA, OGUN STATE, NIGERIA, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF Ph.D DEGREE IN INDUSTRIAL MATHEMATICS (STATISTICS OPTION)

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ACCEPTANCE

This is to attest that this thesis was accepted in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy in Industrial Mathematics (Statistics Option), College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria.

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DECLARATION

I, **OGUNTUNDE Pelumi Emmanuel** (13PCD00573) declare that this research was carried out by me under the supervision of Dr. Adebowale O. Adejumo of the Department of Statistics, University of Ilorin, Ilorin, Nigeria and Dr. Enahoro A. Owoloko of the Department of Mathematics, Covenant University, Ota, Ogun State, Nigeria. I attest that the thesis has not been presented either wholly or partly for the award of any degree elsewhere. All sources of data and schorlarly information used in this thesis are acknowledged accordingly.

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CERTIFICATION

We certify that this thesis titled 'Generalisation of the Inverse Exponential Distribution: Statistical Properties and Applications' is an original work carried out by OGUNTUNDE Pelumi Emmanuel (13PCD00573) in the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria, under the supervision of Dr. Adebowale O. Adejumo and Dr. Enahoro A. Owoloko. We have examined and found the work acceptable for the degree of Doctor of Philosophy in Industrial Mathematics (Statistics Option).

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DEDICATION

To my wife, Omoleye and my beautiful daughter, Emmanuella.

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LIST OF SYMBOLS/ABBREVIATIONS

cdf	Cumulative distribution function
pdf	Probability density function
pmf	Probability mass function
g(x)	The pdf of the baseline distribution
G(x)	The cdf of the baseline distribution
f(x)	The pdf of the compound distribution under consideration
F(x)	The cdf of the compound distribution under consideration
AIC	Akaike Information Criteria
BIC	Bayesian Information Criteria
MLE	Maximum Likelihood Estimation
MSE	Mean Square Error
n	Sample size
X	Random variable
x	Random sample
S(x)	Survival function
h(x)	Hazard function
Q(u)	Quantile function
$\Gamma(.)$	Gamma function
$\gamma(.)$	Incomplete gamma function
$B(a, b)$	Beta function
$I_{G(x)}(a, b)$	Incomplete Beta function ratio

ABSTRACT

There are several real life data sets that do not follow the Normal distribution; these category of data sets are either negatively or positively skewed. However, some could be slightly skewed while others could be heavily skewed. Meanwhile, most of the existing standard theoretical distributions are deficient in terms of performance when applied to data sets that are heavily skewed. To this end, the aim of this study is to extend the Inverse Exponential distribution by inducing it with skewness with a view to enhancing its efficiency. This was achieved by generalising the Inverse Exponential distribution using four different generalised families of distributions. The resulting generalised distributions are: the Kumaraswamy Inverse Exponential (KIE), Transmuted Inverse Exponential (TIE), Exponentiated Generalised Inverse Exponential (EGIE) and Weibull Inverse Exponential (WIE) distributions. Explicit expressions for the densities and basic statistical properties of these compound distributions were established, the method of maximum likelihood estimation (MLE) was used in estimating the unknown parameters and these distributions were fitted to twelve (12) real life data sets to assess their flexibility. R software was adopted and used to perform all the analyses in this study. In addition, a new generalised family of distributions named the Exponential Generalised family of distributions has been defined and explored. Part of the main results in this study include a statistical table that was generated for the Inverse Exponential distribution. A simulation study was also conducted and it was concluded that the standard error and biasedness generated by the parameters of the Exponential Inverse Exponential distribution became small as the sample size increases. Excerpt from the analysis indicates that the derived generalised distributions are more flexible than their sub-models but for the exception of data sets with large variance and outliers. It is recommended that more link functions that will result in generalised families of distributions with only one additional shape parameter and void of special functions should be developed.

Keywords: *Application; Estimation; Generalisation; Inverse Exponential Distribution; Statistical Properties; Simulation.*

CHAPTER ONE

INTRODUCTION

1.1 Preamble

This chapter introduces the basic concepts that led to the development of compound distributions, the statement of the problem, aim and objectives, scope of the study, significance of the study, justification, motivation and limitations.

1.2 Background to the Study

In probability distribution theory, preference for a particular probability distribution in modelling real life phenomena could be based on either the distribution is tractable or the distribution is flexible (Oguntunde, Adejumo, Okagbue and Rastogi, 2016a). The tractability of a probability distribution may be useful in theory because such distribution would be easy to work with; especially when it comes to simulation of random samples, but to practitioners and some other stakeholders, the flexibility of probability distributions could be of interest. In actual fact, it is preferable to make use of probability distributions that best fit the available data set than to transform the existing data set as this may affect the originality of the data set. As a result of this, several efforts have been made in recent years to ensure that the existing standard theoretical distributions are modified and extended (Merovci, 2013a), this could increase their flexibility and enhance the capability to model real life data sets.

To extend an existing standard distribution, there are various approaches that could be adopted. For instance, the flexibility of a distribution can be increased by means of generalisation which involves using the available generalised family of distributions. When a distribution is generalised, extra shape parameter(s) from the family of distributions used would have been added. The role of these additional shape parameter(s) is to vary the tail weight of the resulting compound distribution, thereby inducing it with skewness. The flexibility can also be increased by modifying the existing distribution. For instance, two or more standard distributions can be combined as in the case of convolution, quotient, or product of independent random variables. Also, some distributions are found to be functions of one or more other distributions; the composition of the student t-distribution is a common example (Sun, 2011).

Probability distributions are categorised into either discrete or continuous distributions. Discrete probability distributions are distributions that have random variables which can only assume certain countable values like integers, while a continuous distribution is a type of distribution in which its random variable can assume any value within a specified range; which may be infinite. Discrete distributions are defined by their probability mass function (pmf) while continuous distributions are described by their probability density function (pdf).

There are several standard theoretical distributions in the literature; the Binomial distribution, Poisson distribution, Geometric distribution, Negative Binomial distribution, Exponential distribution, Uniform distribution, Normal distribution, Weibull distribution, Rayleigh distribution, Log-normal distribution, Gamma distribution, Logistic distribution, Frechet distribution, Gumbel distribution, Beta distribution, Kumaraswamy distribution, Pareto distribution, Triangular distribution, Nakagami distribution, Burr distribution, Cauchy distribution, Lomax distribution, Dagum distribution, Gompertz distribution, Chi-square distribution, Student-t distribution, Fisher distribution, Lindley distribution and Birnbaum-Saunders distribution are some notable ones.

While the Binomial, Poisson, Geometric and Negative Binomial distributions represent discrete distributions, the others represent the continuous distributions. These standard distributions have been found to be of immense importance and usage in many fields of study both in theory and practice. For instance, it is common knowledge that the Binomial distribution can be used in situations where the outcome of an event is dichotomous. Some examples of such events include; a student who sat for an examination which can be categorised into either pass or fail, a product from a manufacturer which can be classified as defective or not defective, and so on.

Out of all these standard theoretical distributions, of interest in this study is the Exponential distribution because of some of its attractive properties. For instance, it is the only continuous probability distribution that has the memoryless property (it shares this same property with the Geometric distribution) and it also has a constant failure rate; this indicates that it is suitable for real life situations that have constant risk.

Despite the usage of Exponential distribution in poisson processes, reliability engineering and its attractive properties, the fact that the Exponential distribution has a constant failure rate (Lemonte, 2013) is a disadvantage because for that singular reason, the distribution becomes unsuitable for modelling real life situations with bathtub (uni-antimodal) and inverted bathtub (unimodal) failure rates. This is actually a serious short-coming of the Exponential distribution. Also, the memorylessness is rarely obtainable in real life phenomena. To make up for these limitations, Keller and Kamath (1982) came up with a modified version of the Exponential distribution, this modification resulted into the Inverse Exponential distribution and it has also been studied in some details by Lin, Duran and Lewis (1989).

In recent studies, there have been major and significant advancements in probability distribution theory which involves the introduction of new generalised families of distributions. The works of Cordeiro, Ortega and da Cunha (2013), Bourguignon, Silva and Cordeiro (2014) are some notable ones. These attempts were aimed at developing generalised distributions that would be robust and more flexible than the existing standard theoretical distributions. These generalised distributions are sometimes regarded as compound distributions in this study.

The first study that involves the introduction of a generalised family of distributions from the logit of a random variable is that of Eugene, Lee and Famoye (2002) where the Beta generalised family of distributions was introduced and this has led to the development of Beta Normal distribution. Following this precept, several other generalised distributions have also been introduced. A notable one is the development of the Beta Exponential distribution (Nadarajah and Kotz, 2006) and many more which are mentioned in the next chapter.

In practice, these compound distributions have been found to be better than several standard theoretical distributions in terms of flexibility. Particularly, when a data set is heavily skewed, a generalised distribution tends to fit the data set better than the parent distribution. Therefore, attention has been shifted to favour generalised distributions in recent years (Alshawarbeh, 2011).

In addition to the Beta generalised family of distributions, several other classes or families of generalised distributions have been introduced. For instance, the Kumaraswamy-G family of distributions, Transmuted family of distributions, Gamma-G (type 1) family of distributions, Gamma-G (type 2) family of distributions, Gamma-G (type 3) family of distributions, McDonald-G family of distributions, Log-Gamma-G family of distributions, Exponentiated T-X family of distributions, Exponentiated-G (EG) family of distributions, Logistic-G family of distributions, Gamma-X family of distributions, Logistic-X family of distributions, Weibull-X family of distributions, Weibull-G family of distributions, Marshall-Olkin family of distributions, Beta Marshall-Olkin family of distributions, Kumaraswamy Marshall-Olkin family of distributions and Kumaraswamy Transmuted family of distributions are known families of generalised distributions in the literature (Owoloko, Oguntunde and Adejumo, 2015; Oguntunde *et al.*, 2016a; Cordeiro, Alizadeh, Nascimento and Rasekhi, 2016). Further details about these families of generalised distributions are provided in the next chapter.

Meanwhile, four (4) out of the nineteen different families mentioned above are used and explored in this study because of some of their desirable properties and peculiarities. The four families considered are Kumaraswamy-G family of distributions, Transmuted-G family of distributions, Exponentiated-G family of distributions and Weibull-G family of distributions. The reasons for the selection are:

i The Kumaraswamy-G family was selected because the Kumaraswamy distribution itself shares so many properties in common with the well-known Beta distribution while the Kumaraswamy distribution has more mild algebraic properties. In particular, its cdf excludes special functions like the incomplete beta ratio (Jones, 2009) and it can be used in many areas where the Beta distribution is applicable.

ii The Transmuted-G family was selected because while other families of distributions were formulated using the logit of a random variable, the Transmuted-G family was formulated using a transmutation map; a functional composition of the cdf of a distribution with the quantile function of another (Shaw and Buckley, 2007).

iii The Exponentiated-G family was selected because this form of generalisation has been discovered to be more tractable than the Beta-G family of distributions (Cordeiro *et al.*, 2013; Oguntunde, Odetunmibi and Adejumo, 2015a) and it has mild algebraic properties for simulation purposes because its quantile function has a simple form (Cordeiro *et al.*, 2013). Despite these advantages, this particular family of distributions has not yet been rigorously explored.

iv The Weibull-G family was selected because there exists about two forms of such family in the literature; for instance, the Weibull-X family of distributions and Weibull-G family of distributions are both available in the literature. Besides, the Weibull distribution is the most important distribution for problems in reliability (Bourguignon *et al.*, 2014).

To this end, this study has developed generalisation of the Inverse Exponential distribution with a view to improving its flexibility. After these generalisation, the resulting compound models are Kumaraswamy Inverse Exponential distribution, Transmuted Inverse Exponential distribution, Exponentiated Generalised Inverse Exponential distribution and Weibull Inverse Exponential distribution. These distributions possess additional parameters because of the extra parameter(s) embedded in the families used. Discussed next is the statement of the problem for this research.

1.3 Statement of the Problem

The Exponential distribution is a simple and important distribution in Statistics but some properties that it possesses render it unsuitable for most real life events. For instance, the memoryless property may not be obtainable in the real sense, a typical illustration of such event is given here; if a tyre is purchased and the manufacturer says the probability of the tyre lasting for four years is 0.9, everyone knows that after the first four years, the probability that the same tyre will last for another four years would have reduced either slightly or drastically, but the memoryless property assumes that the probability will still remain the same.

Secondly, it is rare to see an event that has a constant failure rate or risk in real life. For instance, if a man starts a new business, it is expected that the risk will not be constant; the risk might be high at the initial stage and reduce as time goes on. It is then obvious that the Exponential distribution lacks ability to model and describe some important real life events. Hence, attention has been shifted from the Exponential distribution to the Inverse Exponential distribution as also done in this study.

The Inverse Exponential distribution which is a modified version of the Exponential distribution can model data sets with inverted bathtub failure rates (Keller and Kamath, 1982) but its inability to properly model data sets that are highly skewed or that have fat tails has been noticed in the work of Abouammoh and Alshingiti (2009) where the Generalised Inverse Exponential distribution was introduced. Therefore, there is a need to induce skewness into the Inverse Exponential distribution so that it would be able to withstand strong asymmetry in the data sets that are heavily skewed. In addition, some of the families of distributions mentioned earlier involve the Generalised family of distribution as a special case. That means, using these families of distributions in this study will not only extend the Inverse Exponential distribution but also extend the Generalised Inverse Exponential distribution.

To the best of the researcher's knowledge, generalisations that involve extension of the Inverse Exponential distribution are scanty in the literature. Besides, the Beta Inverse Exponential distribution that was proposed by Singh and Goel (2015) involves an incomplete beta function ratio thus, rendering it difficult to handle. Hence, there is a need for other extensions of the Inverse Exponential distribution that would not involve such complex function. This is very important so that the densities of the resulting compound distribution would not look scary and complex.

1.4 Aim and Objectives

Aim: The aim of this study is to develop various extensions of the Inverse Exponential distribution with a view to inducing it with skewness for better modelling especially for heavily skewed data sets.

Objectives: The specific objectives for this study are to:

i generate a concise statistical table for the Inverse Exponential distribution for some selected parameter values;

ii derive generalisations of the Inverse Exponential distribution that would result into Kumaraswamy Inverse Exponential (KIE) distribution, Transmuted Inverse Exponential (TIE) distribution, Exponentiated Generalised Inverse Exponential (EGIE) distribution and Weibull Inverse Exponential (WIE) distribution;

iii establish various statistical properties of the derived distributions;

iv investigate the superiority of the derived distributions over their sub-models using real life data sets; and

v derive a new family of generalised distributions named the 'Exponential-G family of distributions' with its statistical properties. The performance of its parameters are to be assessed through simulation study.

1.5 Scope of the Study

The baseline distribution used in this study is the Inverse Exponential distribution because it is an improvement over the Exponential distribution. The Inverse Exponential distribution is further extended by means of generalisation, other forms of extensions and modifications like convolution, difference, quotient and product are not the focus of this study. In other words, this study is restricted to generalising the Inverse Exponential distribution, stating the statistical properties and applications. Real life data sets which were extracted as secondary data from textbooks and referred journals were used for the analyses. The data sets relate to the fields of engineering, finance, medicine and hydrology.

1.6 Justification for the Research

In real life, events with constant hazard rates, that is, events that do not degrade or ware-out with time are rare. This is why distributions with bathtub or inverted bathtub failure rates are more preferred in reliability engineering because for most complex electronic and mechanical systems, the hazard rates usually exhibit either bathtub or inverted bathtub failure rates (Xie and Lai, 2006; Mansour, Abd-Elrazil, Hamed and Mohamed, 2015). Hence, there is a great need for distributions whose failure rates are not constant in shape.

Dey and Pradhan (2014) noted that though the Inverse Exponential distribution has been applied in several ways, very little has been done on the generalised version of the Inverse Exponential distribution. This implies that the Inverse Exponential distribution has not yet been rigorously explored with respect to the numerous generalised families of distributions that exist in the literature.

Compound distributions with simple densities have the tendency of being preferred in the theory of probability distributions because most of the available family of distributions have two or more additional parameters thus increasing the overall parameters of the resulting compound distributions. In that case, if complex functions are included in the densities, the compound distribution would look scary and complex and would be difficult to handle especially when simulation is to be performed.

1.7 Significance of the study

According to Bourguignon *et al.*, (2014), a number of standard distributions have been widely used over the past decades for modelling real life data sets but in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. Therefore, this research provides generalised distributions derived from an existing baseline distribution and the resulting distributions serves as improvements over the existing baseline distribution. The resulting compound distributions would be useful in the area of distribution theory in Statistics, and in modelling real life data sets relating to biology, demography, economics, environmental sciences, engineering, finance, medicine, hydrology, insurance and other related fields. Also, the resulting compound distributions would be helpful in describing real life phenomena whose failure rate is not constant.

1.8 Motivation

Most compound distributions perform better than their baseline distributions in terms of flexibility when applied to real life data sets. For instance; the Beta-Exponential distribution has been confirmed to perform better when compared with the Exponential distribution by Jafari, Tahmasebi and Alizadeh (2014); the Beta-Nakagami distribution has been confirmed to be a better alternative to the Nakagami distribution when applied to real life data sets by Shittu and Adepoju (2013); the Kumaraswamy-Dagum distribution out-performed the Dagum distribution when applied to a data set on air conditioning system (Huang and Oluyede, 2014); the Transmuted Exponential distribution was confirmed to provide a better fit than the Exponential distribution when applied to real life data sets by Owoloko *et al.*, (2015). Also, the Weibull Exponential distribution was confirmed to provide a better fit when compared with the Exponential distribution (Oguntunde, Balogun, Okagbue and Bishop, 2015b).

Therefore, it becomes interesting that generalising well-known standard distributions can produce more robust distributions. To this end, since Abouammoh and Alshingiti (2009) discovered that the Inverse Exponential has the inability to properly model data sets that are highly skewed or that have fat tails, then, generalising the Inverse Exponential distribution could cater for its shortfalls.

1.9 Limitations of the Research

The compound distributions derived in this study cannot properly model data sets that are symmetric in nature because the derived distributions are skewed. Also, when the variance of the data set is far more than its mean, the distributions are unsuitable.

1.10 Source of Data

The data sets used in this study were collected from secondary sources and particularly, they were extracted from textbooks and referred journals. Details about the data sets are available in Chapter three of this study.

1.11 Definition of Key Terms

The various important statistical terms that are used in this study are defined in this section.

Definition 1.11.1:

Asymmetry: When a distribution is asymmetric in nature, it means the distribution lack symmetry or it is uneven in distribution.

Definition 1.11.2:

Baseline Distribution: It is synonymous to parent distribution. It refers to the existing distribution that is being modified or generalised.

Definition 1.11.3:

Bathtub Shape: It means a shape that have steep sides with a flat bottom. The shape describes a curve that has three parts; decreasing, constant and increasing.

Definition 1.11.4:

Dichotomy: It means division into two contrasting parts or two possible outcomes.

Definition 1.11.5:

Failure Rate: It can also be called hazard function, hazard rate, force of mortality and risk. It is the frequency at which a component or system fails.

Definition 1.11.6:

Inverted Bathtub Shape: This is the opposite of a bathtub shape. Such a curve also has three parts but in this order; increasing, constant and decreasing. The curve would have only one mode or peak.

Definition 1.11.7:

Kurtosis: Kurtosis is the measure of peakedness of a curve or probability distribution. For knowledge sake, kurtosis is of three types; Leptokurtic, Platykurtic and Mesokurtic.

Definition 1.11.8:

Location Parameter: This is a parameter whose change in value could shift the curve either to the left or to the right.

Definition 1.11.9:

Memoryless Property: Literally, memoryless means not having the memory of the past. This is the property in which the future is independent on the past. That is, the system under study is not affected by its history.

Definition 1.11.10:

Outlier: An outlier is an observation that is far distant from the remaining observations in the data set.

Definition 1.11.11:

Scale Parameters: A scale parameter is a parameter in which a change in its values would affect the widening or the shrinking of the plot of the distribution.

Definition 1.11.12:

Shape Parameters: These are parameters that affect the shape of the distribution. A change in the value such parameter could result in an entire change in the shape of the distribution.

Definition 1.11.13:

Skewness: It is a measure of asymmetry of a probability distribution.

Definition 1.11.14:

Symmetric: A distribution that is symmetric in nature is a distribution whose mean, mode and median are the same. An example is the Normal distribution. Such a distribution has zero skewness.

Definition 1.11.15:

Tractable: It means simple. In the context of this study, it could also mean having mild algebraic property (Jones, 2009).

1.12 Summary

In this chapter, various standard distributions have been highlighted and two limitations of the Exponential distribution have been identified. It has been mentioned and emphasised that the Inverse Exponential distribution is the baseline distribution in this study. The focus of this study has been clearly stated and it involves extending the Inverse Exponential distribution for better modelling capability. Besides, four (4) out of the available generalised families of distributions are selected and the reasons for the selection have been clearly stated. As a result, the Kumaraswamy Inverse Exponential distribution, Transmuted Inverse Exponential distribution, Exponentiated Generalised Inverse Exponential distribution and Weibull Inverse Exponential distribution are derived.

In the next chapter, the available families of distributions are discussed extensively and the various works that are related to this present study are also discussed in details. In addition, the gaps in the previous studies are also identified.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Various studies that have been carried out in the past which are related to this present study are discussed in this chapter. About nineteen of the available generalised families of distributions are systematically examined and discussed in this study. Some of these families of distributions have been widely used while some have not received adequate and appreciable attention; this is because some of these families of distributions are relatively new.

2.2 Generalised Families of Distributions

In this section, the various generalised families of distributions available in the literature are discussed extensively. The classes of distributions discussed are the Kumaraswamy-G family of distributions, Transmuted family of distributions, Gamma-G (type 1) family of distributions, Gamma-G (type 2) family of distributions, Gamma-G (type 3) family of distributions, McDonald-G family of distributions, Log-Gamma-G family of distributions, Exponentiated T-X family of distributions, Exponentiated-G (EG) family of distributions, Logistic-G family of distributions, Gamma-X family of distributions, Logistic-X family of distributions, Weibull-X family of distributions, Weibull-G family of distributions, Marshall-Olkin family of distributions, Beta Marshall-Olkin family of distributions, Kumaraswamy Marshall-Olkin family of distributions and Kumaraswamy Transmuted family of distributions. It should be noted that there are several approaches for formulating new families of distributions, these are provided in Table 2.1:

Table 2.1: Families of Distributions

SN	Link Function	Range	Families
1	$G(x)$	(0,1)	Beta-G, Kumaraswamy-G, McDonald-G, Exp-G.
2	$-\log[G(x)]$	$(0, \infty)$	Gamma-G Type II, Log-Gamma-G Type II.
3	$-\log[1 - G(x)]$	$(0, \infty)$	Gamma-G Type I, Log Gamma-G Type I, Weibull-X, Gamma-X.
4	$-\log[1 - G^{\alpha}(x)]$	$(0, \infty)$	Exponentiated T-X.
5	$\frac{G(x)}{1-G(x)}$	$(0, \infty)$	Gamma-G Type III, Weibull-G.
6	$\log\left[\frac{G(x)}{1-G(x)}\right]$	$(-\infty, \infty)$	Logistic-G.
7	$\log\{-\log[1 - G(x)]\}$	$(-\infty, \infty)$	Logistic-X.

Source: Tahir, Cordeiro, Alzaatreh, Mansoor and Zubair (2016)

2.2.1 Beta Generalised Family of Distributions

The Beta Generalised (Beta-G for short) family of distributions was introduced by Eugene *et al.*, (2002). It was derived from the logit of Beta random variable and it has two extra shape parameters.

Starting from an arbitrary cdf, say $G(x)$, the cdf of the Beta-G family of distributions is:

$$F(x) = I_{G(x)}(a, b) \tag{2.1}$$

where $I_{G(x)}(a, b)$ is regarded as the incomplete beta function ratio.

Mathematically:

$$I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw \tag{2.2}$$

Its associated pdf is:

$$f(x) = \frac{1}{B(a,b)} g(x)[G(x)]^{a-1}[1-G(x)]^{b-1} \tag{2.3}$$

for $x > 0, a > 0, b > 0$

where $g(x) = \frac{dG(x)}{dx}$ and $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}dw$
w is an arbitrary random variable, a and b are the additional shape parameters

Following this precept, a number of compound distributions have been derived. For instance, the Beta Normal distribution has been developed by Eugene *et al.*, (2002); this was achieved by making $G(x)$ and $g(x)$ the cdf and pdf of the Normal distribution respectively.

Other distributions that have been derived include Beta Gumbel distribution (Nadarajah and Kotz, 2004); Beta Exponential distribution (Nadarajah and Kotz, 2006); Beta Weibull distribution (Famoye, Lee and Olugbenga, 2005); Beta Rayleigh distribution (Akinsete and Lowe, 2008); Beta Laplace distribution (Kozubowski and Nadarajah, 2008); Beta Pareto distribution(Akinsete, Famoye and Lee, 2008); Beta Generalised Exponential distribution (Barreto-Souza, Santos and Cordeiro, 2010); Beta Nakagami distribution (Shittu and Adepoju, 2013); Beta Inverse Weibull distribution (Hanook, Shahbaz, Moshin and Kibra, 2013); Beta Gompertz distribution (Jafari *et al.*, 2014) and Beta Inverse Exponential distribution (Singh and Geog, 2015).

In the work of Nadarajah and Kotz (2004), the various statistical properties of the Beta Gumbel distribution which include the hazard rate function, asymptotic properties, moments and parameter estimation were discussed. In Nadarajah and Kotz (2006), the moment generating function, characteristic function, shapes, mean deviations, sums and ratios, asymptotics, Renyi and Shannon entropies, estimation of parameters and simulations involving the Beta Exponential distribution were discussed.

General results for the Beta Weibull distribution which involves the special cases, moments and expressions for the moment generating function, asymptotic behaviour, mean deviation, Bonferroni and Lorenz curves, reliability analysis, entropies, parameter estimation using the method of moments and MLE, and applications which illustrates the flexibility of the Beta Weibull distribution over the Weibull distribution were provided in Cordeiro, Nadarajah and Ortega (2012).

In Akinsete and Lowe (2008), the densities and various mathematical properties of the Beta Rayleigh distribution were derived but in Nekoukhou (2015), a discrete version of the Beta Rayleigh distribution named Discrete Beta Rayleigh distribution was developed. Properties of the Discrete Beta Rayleigh distribution which include the survival and hazard function, order statistics, parameter estimation were discussed. A simulation study which investigates the performance of the model parameters was also conducted including an application to a data set representing the results of ten shots fired from a rifle at each of 100 targets. The result showed that the Discrete Beta Rayleigh distribution is more flexible than the other competing distributions like the Discrete Weibull distribution, Discrete Exponential distribution, Exponentiated Discrete Weibull and so on.

For the Beta Laplace distribution that was proposed by Kozubowski and Nadarajah (2008), the focus was on the theoretical properties of the distribution, the moments were derived and the stochastic representations which aid in random variable generation was also provided.

In Akinsete *et al.*, (2008), statistical properties which include the mean, variance, entropies, skewness, kurtosis and mean deviation of the Beta Pareto distribution were derived. Estimation of parameters was also provided and an application to two flood data sets which affirm the superiority of the Beta Pareto distribution over the baseline distribution was presented. The shape of the distribution is unimodal while that of the hazard function could either be decreasing or unimodal.

The Beta Generalised Exponential distribution was discovered to be an extension of the Beta Exponential distribution and the Generalised Exponential distribution (Barreto-Souza *et al.*, 2010). Its properties like the distribution of order statistics, moment generating function, moments and estimation of model parameters were provided. An application to the data set on strength of 1.5cm glass fibre was provided and it was concluded that the Beta Generalised Exponential distribution is more flexible than the Beta Exponential distribution and the Generalised Exponential distribution.

Shittu and Adepoju (2013) developed the densities of the Beta Nakagami distribution including its various properties which include the asymptotics, hazard rate function, moment and moment generating function and estimation of model parameters using the method of MLE. An application to a data set on exceedances of flood peaks between the years 1958 to 1984 was provided and the Beta Nakagami provided a better fit than the Nakagami distribution.

In Hanook *et al.*, (2013), the densities and various statistical properties of the Beta Inverse Weibull distribution were derived. The relationships that exist between its parameters, mean, skewness, variance and kurtosis were also investigated. The shape of the distribution is unimodal and the model parameters were successfully estimated. Though, an application to real life data sets was not provided but the authors claimed that the Beta Inverse Weibull distribution would receive wider attraction in reliability and mechanical engineering.

Jafari *et al.*, (2014) gave the densities and properties of the Beta Gompertz distribution. Distributions like the Exponential distribution, Generalised Exponential distribution, Generalised Gompertz distribution, Gompertz distribution and the Beta Exponential distribution were discovered to be special cases of the Beta Gompertz distribution. A simulation study was conducted to investigate the behaviour of the proposed distribution. An application to a data set on lifetime of 50 devices revealed that the Beta Gompertz distribution has a better fit than all its sub-models.

In a nut shell, Beta generalised distributions are indeed better than their respective baseline distributions when applied to real life data sets due to the various examples stated above. Meanwhile, in theory, these distributions may be difficult to handle. In particular, Barreto-Souza *et al.*, (2010) noted that the resulting pdf; $f(x)$ will be more tractable when $G(x)$ and $g(x)$ have simple analytic expression but if otherwise, $f(x)$ will be difficult to deal with.

2.2.2 Kumaraswamy Generalised Family of Distributions

All the distributions mentioned above would lead to some mathematical difficulties because the cumulative distribution function (cdf) of the Beta distribution is not simple; it involves the incomplete beta function ratio (Cordeiro and de Castro, 2011). To this end, Cordeiro and de Castro (2011) developed and studied another family of generalised distributions based on the Kumaraswamy distribution.

The Kumaraswamy distribution itself was proposed by Kumaraswamy (1980) and it has been identified by Jones (2009) as an alternative and substitute to the well-known Beta distribution. In fact, the Kumaraswamy and Beta distributions share so many properties in common but the major difference is that; the Kumaraswamy distribution does not involve any special function.

Jones (2009) made some comparison between the Kumaraswamy distribution and the Beta distribution. In terms of similarities, the duo have the same general shapes and their shape is dependent on their parameter values, their maximum likelihood estimations (MLE) are similar, they both have simple limiting distributions, their densities behave the same as the value of x approaches zero and as the value of x approaches one, they both have good behaviour of skewness and kurtosis measures, their order statistics from the Uniform distribution can easily be interpreted.

As contained in Jones (2009), the Kumaraswamy distribution has some advantages over Beta distribution. For instance, the distribution function for the Kumaraswamy distribution is much more simpler. Hence, it is easier for computational purposes. Its quantile function is simpler thus making it easier to generate random numbers, the formulae for its moments of order statistics are simpler.

The advantages of Beta distribution over Kumaraswamy distribution are given as follows; Beta distribution has been widely used in Bayesian analysis, the formulae for its moments and the moment generation function (m.g.f) are simpler, Beta distribution can be generated more in physical processes (Jones, 2009). With respect to all these, the Kumaraswamy distribution has been preferred by many researchers because of its mathematical simplicity.

Now, the Kumaraswamy generalised family of distribution is defined as follows: Consider a non-negative random variable X with an arbitrary baseline distribution, say $G(x)$, then the cdf of the Kumaraswamy-G distribution is given as:

$$F(x) = 1 - \{1 - [G(x)]^a\}^b \tag{2.4}$$

The corresponding pdf is given as:

$$f(x) = abg(x)[G(x)]^{a-1} \{1 - [G(x)]^a\}^{b-1} \tag{2.5}$$

for $x > 0, a > 0, b > 0$

where a and b are additional shape parameters

Cordeiro and de Castro (2011) explained that the density function of the Kumaraswamy generalised family of distribution is a time to failure distribution of an entire system.

Some generalised distributions have been defined using the Kumaraswamy generalised family of distribution. For instance, Cordeiro and de Castro (2011) defined the Kumaraswamy Normal distribution, Kumaraswamy Weibull distribution, Kumaraswamy Gamma distribution, Kumaraswamy Gumbel distribution and Kumaraswamy Inverse Gaussian distribution. General expansion for the density function, general formulae for the moments and distribution of order statistics were also provided. An application to a data set on adult number of T. confusum cultured at 29^0C which was provided by Eugene *et al.*, (2002) showed that the Kumaraswamy Normal distribution performs almost the same as the Beta Normal distribution and better than Kumaraswamy Exponential, Kumaraswamy Gamma and Gamma distributions.

In Nadarajah and Eljabri (2013), the densities and statistical properties of the Kumaraswamy Generalised Pareto distribution were derived. The properties considered include the shape of the distribution and that of the hazard function, the moments, distribution of order statistics, the extreme values and estimation of parameters. A simulation study was conducted to assess the performance of the parameter estimates and an application to a data set on the exceedances of the threshold $65m^3s^{-1}$ by a river called River Nidd at Hunsingore Weir between the years 1934 to 1969 revealed that the Kumaraswamy Generalised Pareto distribution provides a better fit than the Exponentiated Generalised Pareto distribution and the Generalised Pareto distribution.

In Shahbaz, Shahbaz and Butt (2012), the densities and statistical properties of the Kumaraswamy Inverse Weibull distribution were derived. The properties derived include the moments, quantile function, hazard function and estimation of model parameters. An application to two different data sets which relate to data on survival time and data on salary was provided and it was shown that the Kumaraswamy Inverse Weibull distribution provides a better fit than the Inverse Weibull distribution, Weibull distribution and Gamma distribution.

The Kumaraswamy-Dagum distribution was defined by Huang and Oluyede (2014) and its exponentiated version was studied in much details. An application to the data sets relating to air conditioning system, basket ball player salary and poverty rate revealed that the Kumaraswamy Dagum distribution is more flexible than the Dagum distribution, Exponentiated Kumaraswamy Weibull distribution and the Beta Kumaraswamy Weibull distribution.

El-Batal and Kareem (2014) defined the five-parameter Kumaraswamy Exponentiated Lomax distribution, its special cases include the Exponentiated Lomax distribution, Kumaraswamy Exponentiated Pareto distribution, Exponentiated Pareto distribution, Kumaraswamy Pareto distribution, Kumaraswamy Lomax distribution and Lomax distribution. Statistical properties of the Kumaraswamy Exponentiated Lomax distribution which include the quantile function, moments, distribution of order statistics and moment generating function were derived including estimation of model parameters.

Al-Babtain, Fattah, Ahmed and Merovci (2015) defined a seven-parameter distribution called the Kumaraswamy-Transmuted Exponentiated modified Weibull distribution. The distribution is so flexible that it has about fifty-four other distributions as special cases. Examples of the sub-models include; Kumaraswamy Transmuted Weibull distribution, Kumaraswamy Transmuted Rayleigh distribution, Exponentiated Transmuted Exponential distribution, Exponentiated Exponential distribution and many others. Its properties like the moments and quantile function were provided including estimation of model parameters. An application to the data set that relates to the nicotine measurements gotten from several brands of cigarettes showed that the Kumaraswamy-Transmuted Exponentiated modified Weibull distribution is more flexible than the Transmuted Exponentiated modified Weibull distribution, Transmuted Exponentiated Rayleigh distribution, Transmuted Rayleigh distribution and Rayleigh distribution. An application to another data set relating to the Greenwich data for non-nested models showed that the Kumaraswamy-Transmuted Exponentiated modified Weibull distribution is more flexible than the Kumaraswamy modified Weibull distribution, Beta modified Weibull distribution, Beta Generalised Exponential distribution, Weibull Pareto distribution, Weibull Geometric distribution, Gamma Lomax distribution, Generalised Gamma distribution, Extended Weibull distribution, Quasi Lindley distribution and Lindley distribution.

Oguntunde, Odetunmibi, Okagbue, Babatunde and Ugwoke (2015c) derived the densities of the Kumaraswamy Power distribution. Its special cases include the Power distribution, Generalised Power distribution and the Exponentiated Power distribution. Various mathematical properties which include the hazard function, survival function, moments and estimation of model parameters were provided but applications to real life data sets were not considered.

Merovci and Elbatal (2015a) provided the densities for the four-parameter Kumaraswamy Linear Exponential distribution. Statistical properties like the moments, moment generating function and hazard rate function were derived including the estimation of model parameters.

Chukwu and Ogunde (2016) studied the Kumaraswamy Gompertz Makeham distribution, it has five parameters and the Gompertz Makeham distribution was identified as its special case. Statistical properties like the asymptotic behaviour, reliability function, hazard rate function, distribution of order statistics and estimation of parameters were studied. An application to a data set that relates to the breaking stress of carbon fibres confirmed that the Kumaraswamy Gompertz Makeham distribution performs better than the Gompertz Makeham distribution, Zografos-Balakrishnan Log-logistic distribution, Beta Frechet distribution and Kumaraswamy Pareto distribution.

From the various examples that have been considered above, it can be concluded that the Kumaraswamy generalised family of distribution is flexible and has simpler algebraic composition. Hence, it has received wider applications in distribution theory in recent times.

2.2.3 Transmuted Family of Distributions

The Transmuted family of distributions was developed by Shaw and Buckley (2007); they were prompted by the need to provide parametric families of distribution that would be flexible and at the same time tractable. The generalised distributions that would result from the use of this form of generalisation are expected to be useful not only in finance but in other wider areas in Statistics.

The formulation of the transmuted family of distribution involves the use of transmutation map. The transmuted map was described by Shaw and Buckley (2007) as a function which comprises the cdf of one distribution with the quantile function of another. This approach was aimed at inducing skewness or kurtosis, as the case may be, into available distributions. Recently, Bourguignon, Ghosh and Cordeiro (2016) extended the work of Shaw and Buckley (2007) to include general results, bivariate and multivariate generalisation.

Several distributions have been generalised using this same approach. For instance, the Transmuted Weibull distribution was developed by Aryal and Tsokos (2011), its densities were derived, and mathematical properties which include the moments and quantiles, reliability function and hazard rate function, cumulative hazard rate function, distribution of order statistics and parameter estimation were derived and established. The Transmuted Exponential distribution and the Exponential distribution were found as special cases. An application to a data set that relates to the tensile fatigue characteristics of a polyster/viscose yarn revealed that the Transmuted Weibull distribution provides a better fit than the Exponentiated Weibull distribution and the Weibull distribution.

In Merovci (2013b), the densities of the Transmuted Rayleigh distribution were provided and statistical properties like the moments, parameter estimation, reliability function, hazard rate function and distribution of order statistics were investigated. The Transmuted Rayleigh distribution was applied to a data set that relates to the nicotine measurements gotten from several brands of cigarette and it was clearly discovered that the Transmuted Rayleigh distribution fits the data better than the Rayleigh distribution.

Ashour and Eltehiwy (2013a) developed the Transmuted Exponentiated Modified Weibull distribution, its densities were derived and some important models like the Exponentiated Modified Weibull distribution, Modified Weibull distribution, Exponentiated Weibull distribution, Transmuted exponentiated Weibull distribution, Transmuted Weibull distribution, Exponentiated Exponential distribution, Transmuted Exponentiated Exponential distribution, Exponential distribution, Weibull distribution, Transmuted Exponential distribution, Generalised linear failure rate distribution, Transmuted generalised linear failure rate distribution, Linear failure rate distribution, Transmuted linear failure rate distribution, Generalised Rayleigh distribution, Transmuted Generalised Rayleigh distribution, Transmuted Rayleigh distribution and Rayleigh distribution are its special cases. In addition, statistical properties which include the reliability function, hazard rate function, quantile function, moments, parameter estimation and distribution of order statistics were established. It is expected that this distribution would be very flexible because it generalises a number of other distributions but an application to a real life data set that would support this was not provided.

In Khan and King (2013), the densities of the Transmuted Modified Weibull distribution were derived and eleven distributions which include the Transmuted Modified Exponential distribution, Transmuted Modified Rayleigh distribution, Modified Weibull distribution, Modified Rayleigh distribution, Modified Exponential distribution, Transmuted Weibull distribution, Transmuted Rayleigh distribution, Transmuted Exponential distribution, Weibull distribution, Rayleigh distribution and Exponential distribution. Properties like reliability function, hazard rate function, quantile, moments, moment generating function, distribution of order statistics and parameter estimation were explored. A practical illustration using the data set that relates to the maximum flood levels was provided and it was confirmed that the Transmuted Modified Weibull distribution is better than than its competing models.

In Ashour and Eltehiwy (2013b), the densities of the Transmuted Lomax distribution were developed. It was discovered that the Lomax distribution and the Transmuted Pareto distribution are its sub-models, properties like the moments, quantiles, mean deviation, reliability function, hazard rate function, cumulative hazard rate function, estimation of parameters and distribution of order statistics were also established. It can be noted that an application to real life data set was not provided and the potentials of the Transmuted Lomax distribution was not investigated.

In Hussian (2014), the densities of the Transmuted Exponentiated Gamma distribution were developed, the Exponentiated Gamma distribution and Gamma distribution are its sub-models. Properties like the reliability function, hazard rate function, quantile function, moments, distribution of order statistics and estimation of parameters were examined. A simulation study was conducted at different true parameter values for $n = 25, 50, 75$ and 100. The estimates of the true parameters were obtained and their respective mean square error (MSE). It was discovered that the mean square errors decreases as the sample size increases.

In Ahmad, Ahmad and Ahmed (2014), the densities of the Transmuted Inverse Rayleigh distribution were derived and the Inverse Rayleigh distribution was identified as its special case. Its statistical properties like the moments, moment generating function, quantiles, estimation of parameters, reliability function, hazard rate function and distribution of order statistics were also established. Though, it was claimed that the Transmuted Inverse Rayleigh distribution can be used to analyse more complex data sets but a real life application to support the claim was not provided.

In Merovci and Puka (2014), the densities of the Transmuted Pareto distribution were derived and Pareto distribution was identified to be its special case. Its statistical properties like the moments, moment generating function, quantiles, estimation of parameters, reliability function, hazard rate function and distribution of order statistics were also established. An application to the data set that relates to the exceedances of flood peaks was provided and it was confirmed that the Transmuted Pareto distribution is better than the Pareto distribution. Though, the Transmuted Pareto distribution failed to perform better than the Generalised Pareto distribution and the Exponentiated Weibull distribution based on their log-likelihood values. A similar result was also obtained when a data set that relates to the annual flood discharge rate was used.

Khan, King and Hudson (2014) developed the densities of the Transmuted Inverse Weibull distribution. It was discovered that its sub-models include the Transmuted Inverse Rayleigh distribution, Transmuted Inverse Exponential distribution and the Weibull distribution. Statistical properties like the moments, mean (geometric and harmonic), mean deviation, entropy, distribution of order statistics and estimation of parameters. Also, the relationships that exist between the shape parameter, pdf, cdf, percentile life, reliability function, hazard rate function, mean, variance, coefficient of skewness, coefficient of kurtosis, and coefficient of variation were presented. It was discovered that the Transmuted Inverse Weibull distribution is capable and suitable for modelling various shapes of aging and failure criteria.

In Elbatal (2013), the densities of the Transmuted Modified Inverse Weibull Distribution were developed and the special cases include the Transmuted Inverse Weibull distribution, Transmuted Modified Inverse Rayleigh distribution and the Transmuted Modified Inverse Exponential distribution. Statistical properties like the reliability function, hazard rate function, quantile function, moments, moment generating function, distribution of order statistics and estimation of parameters were established. Although, a real life application to illustrate the usefulness and potentials of the Transmuted Modified Inverse Weibull distribution was not provided.

The densities of the Transmuted Gompertz distribution was developed by Abdul-Moniem and Seham (2015) and the Gompertz distribution was identified as its submodel. Statistical properties which include the reliability function, hazard rate function, reversed hazard rate function, moments, distribution of order statistics and estimation of parameters were derived and established. A real life application to the data set that relates to the life of fatigue fracture of Kevlar 373/epoxy that are subjected to constant pressure at the 90 percent stress level until all had failed was presented and it was illustrated that the Transmuted Gompertz distribution is better than the Gompertz distribution based on their log-likelihood values.

Afify, Hamedani, Ghosh and Mead (2015) derived the densities of the Transmuted Marshall-Olkin Frechet distribution, it has four parameters and its special cases include the Transmuted Marshall-Olkin Inverse Rayleigh distribution, Transmuted Marshall-Olkin Inverse Exponential distribution, Transmuted Inverse Exponential distribution, Marshall-Olkin Frechet distribution, Transmuted Inverse Rayleigh distribution, Transmuted Frechet distribution, Frechet distribution, Inverse Rayleigh distribution and the Inverse Exponential distribution. Statistical properties like the moments, residual life function, reversed residual life function, quantiles, generating functions, entropies, distribution of order statistics, characterizations and estimation of parameters were studied. An application to two real life data sets relating to breaking stress of carbon fibres and strengths of 1.5cm glass fibres were presented and the Transmuted Marshall-Olkin

Frechet distribution was confirmed to provide better fits than the Beta Frechet distribution, Generalised Exponential Frechet distribution, Marshall-Olkin Frechet distribution, Transmuted Frechet distribution and Frechet distribution.

Owoloko *et al.*, (2015) developed the Transmuted Exponential distribution and the Exponential distribution was identified as its sub-model. Statistical properties like the moments, quantile function, reliability function, hazard rate function and estimation of parameters were studied. An application to a data set that relates to the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90 percent stress level until all had failed confirmed that the Transmuted Exponential distribution is more flexible than the Beta Exponential distribution, Generalised Exponential distribution and the Exponentiated Exponential distribution. Another application to the data set that relates to the monthly actual taxes revenue also confirmed the same result.

Bourguignon *et al.*, (2016) extended the Transmuted family of distributions to include general results like the asymptotes and shapes, quantile function, moments, moment generating function, mean deviation, distribution of order statistics, information theory, bivariate and multivariate generalisation and parameter estimation. An application to the data set that relates to anxiety among a group of 166 normal women was presented and the Transmuted Kumaraswamy distribution was confirmed to provide a better fit than the Kumaraswamy distribution.

These notable examples in the literature are indicators that the Transmuted family of distribution is versatile and flexible as most of the applications provided revealed that generalisation involving the Transmuted family of distribution provides better fits than their counterpart distributions.

Starting from an arbitrary parent cdf; $G(x)$, a non-negative random variable X is said to have a transmuted distribution if its cdf is given as:

$$F(x) = (1 + \lambda)\, G(x) - \lambda[G(x)]^2 \tag{2.6}$$

The corresponding pdf is given as:

$$f(x) = g(x)\left[1 + \lambda - 2\lambda G(x)\right] \tag{2.7}$$

where $|\lambda| \leq 1$

It is good to note that if $\lambda = 0$, then the densities of the Transmuted generalised family of distributions reduces to the densities of the parent distribution.

2.2.4 Exponentiated Generalised Family of Distributions

The Exponentiated Generalised (EG) family of distributions was defined by Cordeiro *et al.*, (2013). It was said that this form of generalisation is an extension of the Exponentiated type distribution. Also, generalised distributions derived from this form of generalisation could be useful in biology, engineering and medicine.

For a continuous random variable X, the cdf of the Exponentiated Generalised (Exponentiated-G) class of distribution is defined as:

$$F(x) = \left\{1 - \left[1 - G(x)\right]^{a}\right\}^{b} \tag{2.8}$$

The corresponding pdf is given as:

$$f(x) = abg(x)\left[1 - G(x)\right]^{a-1}\left\{1 - \left[1 - G(x)\right]^{a}\right\}^{b-1} \tag{2.9}$$

where $a > 0, b > 0$ are additional shape parameters.

According to Cordeiro *et al.*, (2013), the cdf of the Exponentiated Generalised family of distributions is more tractable than that of Beta-G family of distributions since the former excludes special functions like the incomplete beta function ratio. It was also claimed that the lifetime of a device could follow the Exponentiated Generalised family of distributions (Cordeiro *et al.*, 2013).

With this understanding, the Exponentiated Generalised Frechet distribution, Exponentiated Generalised Normal distribution, Exponentiated Generalised Gamma distribution, Exponentiated Generalised Gumbel, Exponentiated Generalised Exponential distribution, Exponentiated Generalised Logistic distribution, Exponentiated Generalised Pareto distribution and Exponentiated Generalised Beta distribution were all introduced by Cordeiro *et al.*, (2013). General properties like the expansion of densities, moments, generating function, mean deviations, distribution of order statistics and estimation of parameters were discussed. These distributions were fitted to four real life data sets which relate to Ethylene data, Wheaton River data, Stress level data and breaking strength of Carbon fibre. An application to the Ethylene data set revealed that the Exponentiated Generalised Normal distribution provides a better fit than the Exponentiated Normal distribution, Lehmann II Normal distribution and Normal distribution.

The illustration made with the Wheaton River data set confirmed the superiority of the Exponentiated Generalised Gumbel distribution over the Exponentiated Gumbel distribution, Lehmann II Gumbel distribution and Gumbel distribution. The Exponentiated Generalised Frechet distribution fits the data set on stress level than the Exponentiated Frechet distribution, Lehmann II Frechet distribution and Frechet distribution while the Exponentiated Generalised Gamma distribution performs better than the Exponentiated Gamma distribution, Lehmann II Gamma distribution and Gamma distribution on application to the carbon data set.

Also, the Exponentiated Generalised Inverse Weibull distribution was developed by Elbatal and Muhammed (2014) and the Exponentiated Inverse Exponential distribution, Generalised Inverse Weibull distribution, Inverse Weibull distribution and Inverse Exponential distribution can be identified as its special cases. The quantile function, moments, moment generating function, distribution of order statistics and estimation of parameters were studied. An application to three real data sets which relate remission times (in months) of bladder cancer patients, lifetimes of devices, and data on patients suffering from leukemia was provided. In all these three applications, it was shown that the Exponentiated Generalised Inverse Weibull distribution provides better fits than the Generalised Inverse Weibull distribution and the Inverse Weibull distribution.

In Andrade, Rodrigues, Bourguignon and Cordeiro (2015), the densities of the Exponentiated Genaralised Gumbel distribution were derived. The Exponentiated Gumbel distribution and the Gumbel distribution were identified as its special cases. Its shape was established, the quantile function, moments, generating function, mean deviations, Renyi Entropy, distribution of order statistics and estimation of parameters were also derived. A Monte Carlo simulation was conducted using 10,000 replications, random samples of size $n = 100, 200, 400$ and 800 were considered at selected parameter values. It was concluded that the bias and mean square error decrease for all cases as the sample size increases. An application to a data set that relates to the values of March precipitation confirmed that the Exponentiated Generalised Gumbel distribution provides better fits than the Beta Gumbel distribution and the Kumaraswamy Gumbel distribution. Another real life application to the data set relating to the time between failures for repairable item also gave the same result and conclusion.

In Oguntunde *et al.*, (2015a), the densities of the Exponentiated Generalised Weibull distribution were derived and the Generalised Weibull distribution, Exponentiated Weibull distribution, Weibull distribution and Exponential distribution were identified as its special cases. Various properties which include the limiting behaviour, reliability function, hazard rate function, moments, quantile function, distribution of order statistics and estimation of parameters were derived. An application to real life data sets was not provided and the potentials of the Exponentiated Generalised Weibull distribution over other models was not investigated.

The Exponentiated Generalised Frechet distribution was explored by Abd-Elfattah, Assar and Abd-Elghaffar (2016). Its shape was established, its asymptotic behaviour, reliability function, hazard rate function, quantile function, moments, moment generating function, distribution of order statistics and estimation of parameters were also studied. Meanwhile, an application to real life data sets was not considered.

In addition, the Exponentiated Generalised Exponential distribution was explored in Oguntunde, Adejumo and Adepoju (2016b). Important lifetime models like the

Generalised Exponential distribution, Exponentiated Exponential distribution and Exponential distribution were discovered to be its sub-models. Properties like the asymptotic behaviour, reliability function, hazard rate function, moments, generating functions, distribution of order statistics and estimation of parameters were studied in details. An application to real data sets which relate the remission times of bladder cancer patients and death times of patients with cancer of tongue confirms the superiority of the Exponentiated Generalised Exponential distribution over the Generalised Exponential distribution, Exponentiated Exponential distribution and Exponential distribution.

From all these, it can be said that the Exponentiated Generalised family of distribution has some potentials in terms of modelling real life phenomena. Meanwhile, more works need to be done as it has not been rigorously explored like some other family of distributions.

2.2.5 Weibull Generalised Family of Distributions

Two forms of the Weibull Generalised family of distributions are available in the literature. One was named Weibull-G family of distributions while the other was named Weibull-X family of distributions. One common attribute of the Weibull-G and Weibull-X families is that they were both generated from the Weibull random variables.

Weibull distribution in itself is very versatile, therefore, it is expected that generalised distributions derived from these form of generalisations would be versatile and flexible.

To start with, the cdf of the Weibull distribution with positive parameters α and β is:

$$F(x) = 1 - e^{-\alpha x^{\beta}} \tag{2.10}$$

where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a scale parameter.

Now, replacing x with $\frac{G(x)}{1-G(x)}$, then the cdf of the Weibull-G family is:

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right\} \tag{2.11}$$

Of course, it can be derived using the transformation:

$$F(x) = \int_{0}^{\frac{G(x)}{1-G(x)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta-1}} dt \tag{2.12}$$

The corresponding pdf of the Weibull-G family of distributions is obtained by differentiating Equation (2.11) with respect to x. Hence,

$$f(x) = \alpha\beta g(x)\frac{[G(x)]^{\beta-1}}{[1-G(x)]^{\beta+1}} \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right\} \tag{2.13}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters.

Remark: The resulting compound distributions from this generalisation can involve both shape, scale and location parameters depending on the parameters of the parent distribution.

Based on this approach, the Weibull-Uniform distribution, Weibull-Weibull distribution, Weibull-Burr XII distribution and Weibull-Normal distribution have been introduced by Bourguignon, Silva and Cordeiro (2014). General properties which include the distribution of order statistics and estimation of model parameters have been extensively studied. An application to the data set that relates to strengths of 1.5 cm glass fibres confirms the superiority of the Weibull Exponential distribution over the Exponentiated Weibull distribution and Exponentiated Exponential distribution. Another application to the data set relating to fatigue time of aluminum coupons indicated that the Weibull Burr XII distribution provides a better fit than the Beta Burr XII distribution.

Also, Merovci and Elbatal (2015b) derived the densities of the Weibull Rayleigh distribution. It has three parameters and its properties like the quantile function, moments, distribution of order statistics and estimation of parameters were studied. An application to the data set that relates to the failure and running times for a sample of devices from a eld-tracking study of a larger system confirmed that the Weibull Rayleigh distribution provides better fits than the Beta Weibull distribution, Exponentiated Weibull distribution and Weibull distribution.

In Oguntunde *et al.*, (2015b), the densities of the Weibull Exponential distribution were studied in more details, the Gompertz distribution, Exponential-Exponential distribution and the Exponential distribution were identified as its special cases. Various statistical properties like the reliability function, hazard rate function, limiting behaviour, moments, distribution of order statistics and estimation of parameters were studied. An application to a real life data set which relates to the breaking stress of carbon fibres of 50 mm length confirmed that the Weibull Exponential distribution provides better fits than the Exponential distribution. Another application to a data set relating to the strengths of 1.5 cm glass fibres also confirms the same result.

From all these results, it can be said that the Weibull-G family of distribution is flexible, this is expected because the Weibull distribution itself is flexible and versatile. On the other hand, the Weibull-X family of distributions was defined by Alzaatreh *et al.*, (2013) using the transformation:

$$F(x) = \int_0^{-\log[1-G(x)]} g(t)dt \tag{2.14}$$

where $g(t)$ is the Weibull distribution and T is a random variable. That is:

$$g(t) = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1} e^{-\left(\frac{t}{\beta}\right)^{\alpha}} \qquad ; \qquad t \geq 0 \tag{2.15}$$

The cdf of the Weibull-X family of distributions is given as:

$$F(x) = 1 - \exp\left\langle -\left\{\frac{\log[1-G(x)]}{\beta}\right\}^{\alpha}\right\rangle \tag{2.16}$$

The corresponding pdf is:

$$f(x) = \frac{\alpha g(x)}{\beta[1-G(x)]}\left\{\frac{-\log[1-G(x)]}{\beta}\right\}^{\alpha-1} \exp\left\langle -\left\{\frac{\log[1-G(x)]}{\beta}\right\}^{\alpha}\right\rangle \tag{2.17}$$

where $\alpha > 0$ and $\beta > 0$ are additional shape parameters.

This approach was used to define the Weibull-Pareto distribution in Alzaatreh *et al.*, (2013). Its properties like the limiting behaviour, reliability function, hazard rate function, moments, moment generating function, quantile function, mode, mean deviation, distribution of order statistics and estimation of parameters were studied. An application to a data set that relates to the exceedances of flood peaks confirmed that the Weibull Pareto distribution provides a better fit than the Transmuted Pareto distribution and Kumaraswamy Pareto distribution. Meanwhile, much work has not been done in this regard to establish this claim.

It can also be observed that the densities of the Weibull-G family of distributions appear simpler than that of the Weibull-X family of distributions.

2.2.6 Gamma-G (Type I) Family of Distributions

The Gamm-G (Type I) family of distributions was derived by Zografos and Balakrishnan (2009). Therefore, it was named as Zografos-Balakrishnan-G distribution. It was generated from Gamma random variables.

The cdf of the Zografos-Balakrishnan-G family of distributions was derived from the transformation:

$$F(x) = \frac{1}{\Gamma(a)} \int_0^{-\log[1-G(x)]} t^{a-1} e^{-t} dt \tag{2.18}$$

Hence, it can be given as:

$$F(x) = \frac{\gamma\left(a, -\log[1-G(x)]\right)}{\Gamma(a)} \tag{2.19}$$

for $x > 0, a > 0$

where $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$ which is the gamma function and $\gamma(a,k) = \int_0^k t^{a-1}e^{-t}dt$ is the incomplete gamma function.

The pdf of the Gamma (Type I) family of distributions is:

$$f(x) = \frac{1}{\Gamma(a)} g(x) \{-log[1 - G(x)]\}^{a-1} \tag{2.20}$$

for $x > 0, a > 0$.

where a is an additional shape parameter.

When $a = 1$, the expression in Equation (2.20) reduces to give the density of the parent distribution.

With this knowledge, the densities of the Zografos-Balakrishnan Log-logistic distribution were derived by Ramos, Cordeiro, Marinho, Dias and Hamadani (2013). The Log-logistic distribution was identified as its special case. Its properties which include the moments, quantile function, generating function, Renyi Entropy, reliability analysis, characterisations, distribution of order statistics and estimation of parameters were studied in details. An application to a data set that relates to the survival times of patients with breast cancer was presented and the Zografos-Balakrishnan Log-logistic distribution was confirmed to provide a better fit than the Kumaraswamy Log-Logistic distribution, Beta Log-Logistic distribution and Exponentiated Log-Logistic distribution. Although, it did not perform better than the Exponentiated Weibull distribution and the possible reason(s) for this was not stated.

Also, this particular family of distribution has not been rigorously explored like others.

2.2.7 Gamma-G (Type II) Family of Distributions

The Gamma-G (Type II) family of distribution was defined by Ristic and Balakrishnan (2012). In recent time, it has been regarded to as Ristic-Balakrishnan-G family of distribution. Like the Gamma-G (Type I) family of distribution, it was also generated from Gamma random variables.

According to Ramos *et al.*, (2013), its survival function was generated using the transformation:

$$S(x) = \frac{1}{\Gamma(a)} \int_0^{-\log[G(x)]} t^{a-1} e^{-t} dt \qquad ; \qquad x > 0, a > 0 \tag{2.21}$$

Its pdf is:

$$f(x) = \frac{1}{\Gamma(a)} g(x) \{-log[G(x)]\}^{a-1} \tag{2.22}$$

for $x > 0, a > 0$

where a is an additional shape parameter.

This form of generalisation has been used to derive the Gamma Exponentiated Exponential distribution by Ristic and Balakrishnan (2012). The Exponentiated Exponential distribution was clearly identied as one of its special cases, its statistical properties were derived and studied and estimation of parameters was presented. Application to real life data sets were presented but much details about this are not readily accessible.

Recently, the Gamma Exponentiated ExponentialWeibull distribution has been developed by Pogany and Saboor (2016), it has four parameters and its properties like the moments, quantile function and estimation of parameters were discussed. Applications to two real life data sets relating to data on carbon fibres and data on cancer patients were provided. The Gamma Exponentiated ExponentialWeibull distribution was confirmed to provide a better fit than the Extended Weibull distribution, ExponentialWeibull distribution, Gamma Exponentiated Exponential distribution, Weibull distribution and Gamma distribution in both illustrations.

There also exist Gamma-G (Type III) family of distribution defined by Torabi and Montozari (2012) but much information about it is not accessible.

2.2.8 McDonald Generalised Family of Distributions

The McDonald distribution in itself is synonymous with the Generalised Beta distribution of the first kind (GB1). Therefore, the McDonald-G family of distribution was generated from the McDonald random variables by Alexander, Cordeiro, Ortega and

Sarabia (2012).

The pdf of the McDonald distribution is:

$$f(x) = \frac{c}{B(ac^{-1}, b)} x^{a-1} \left(1 - x^c\right)^{b-1} \tag{2.23}$$

for $0 < x < 1$

where $a > 0, b > 0, c > 0$ are shape parameters.

Meanwhile, if $c = 1$, $f(x)$ in Equation (2.23) reduces to the well-known Beta distribution whose pdf is:

$$f(x) = \frac{1}{B(a, b)} x^{a-1} \left(1 - x\right)^{b-1} \tag{2.24}$$

for $0 < x < 1$

where $a > 0, b > 0$ are shape parameters.

Now, the cdf of the McDonald-G family of distributions is:

$$F(x) = I_{[G(x)]^c}\left(ac^{-1}, b\right) \tag{2.25}$$

for $x > 0, a > 0, b > 0, c > 0$

where a, b and c are additional shape parameters.

$$I_{[G(x)]^c}\left(ac^{-1}, b\right) = \frac{1}{B(ac^{-1}, b)} \int_0^{[G(x)]^c} w^{ac^{-1}-1} \left(1 - w\right)^{b-1} dw \tag{2.26}$$

The corresponding pdf is:

$$f(x) = \frac{c}{B(ac^{-1}, b)} g(x) [G(x)]^{a-1} \left\{1 - [G(x)]^c\right\}^{b-1} \tag{2.27}$$

for $x > 0, a > 0, b > 0, c > 0$

where a, b and c are additional shape parameters.

This form of generalisation has been used by Cordeiro, Cintra, Rego and Ortega (2012) to obtain the densities of the McDonald Normal distribution. It has five parameters and its shapes were established. Properties like the moments, generating function, mean deviations, distribution of order statistics, asymptotic behaviour, hazard rate function, Shannon entropy and estimation of parameters were carefully derived. An application to three different real life data sets which relate to data on plasma, skin folds and mass was provided and the McDonald Normal distribution provides a better fit than the Beta Normal distribution, Kumaraswamy Normal distribution, Exponentiated Normal distribution, Normal distribution and Skew Normal distribution.

Tahir, Mansoor, Zubair and Hamedani (2014) derived the McDonald Log-logistic distribution, its properties like the reliability function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, moments, quantile function, mode, mean residual life, entropies, estimation of parameters and characterisations were studied in details. The Beta Log-logistic distribution, Kumaraswamy Log-logistic distribution, Generalised Log-logistic distribution, Exponentiated Log-logistic distribution and Log-logistic distribution were discovered to be its special cases. The distribution was applied to a data set which relates to the survival times of patients with breast cancer and it was confirmed to provide a better fit than the McDonald Weibull distribution, Zografos-Balakrishnan Log-logistic distribution, Beta Log-logistic distribution, Log-logistic distribution, Kumaraswamy Log-logistic distribution, Gamma distribution and the Log-normal distribution.

The McDonald Gompertz distribution has also been defined by Roozegar, Tahmasebi and Jafari (2015) It has five parameters. The Kumaraswamy Gompertz distribution, Beta Gompertz distribution, Beta Generalised Exponential distribution, Beta Exponential distribution, Generalised Gompertz distribution, Kumaraswamy Exponential distribution, Generalised Exponential distribution, Gompertz distribution and Exponential distribution were all identified as its special cases. Its moments, moment generating function, distribution of order statistics, quantiles, entropies and estimation of parameters were discussed and studied. It was fitted to data sets representing the lifetimes of 50 devices and data set on strengths of 1.5 cm glass fibers, the results confirmed the superiority of the McDonald Gompertz distribution over the Beta Gompertz distribution, Kumaraswamy Gompertz distribution and McDonald Exponential distribution.

The McDonald Quasi Lindley distribution was derived by Roozegar and Esfandiyari (2015). It has five parameters which are classified into shape and scale parameters. The Beta Quasi Lindley distribution, Kumaraswamy Quasi Lindley distribution, McDonald Lindley distribution Beta Lindley distribution, Generalised Quasi Lindley distribution, Generalised Lindley distribution and the McDonald Gamma (McG) distribution were all identified as its special cases. Properties like the moments, moment generating function, distribution of order statistics, entropies amd estimation of parameters were studied. The McDonald Quasi Lindley distribution was fitted to a data set which relates to the time between failures of secondary reactor pumps and it was confirmed to provide a better fit than the McDonald Dagum distribution, McDonald Weibull distribution, McDonald Log-logistic distribution, Kumaraswamy Quasi Lindley distribution, Beta Quasi Lindley distribution.

Another form of the McDonald Quasi Lindley distribution was explored by Merovci, Elbatal and Puka (2015). Properties like the reliability function, hazard rate function, reversed hazard rate function, quantile function, moments, moment generating function, distribution of order statistics and estimation of parameters were derived. It was fitted to a data set which relates to the breaking stress of carbon fibres and it provides a better fit than the Quasi Lindley distribution and Lindley distribution. However, the major difference noticed between the works of Roozegar and Esfandiyari (2015) and that of Merovci *et al.*, (2015) is in the type of data sets used and the distributions used as basis for comparison.

2.2.9 Log-Gamma Generalised Family of Distributions

The Log-Gamma-G family of distribution was developed by Amini, MirMostafaee and Ahmadi (2014). In particular, it was developed by the application of the inverse probability integral transformation to the log-gamma distribution.

The densities of the Log-Gamma-G family of distributions according to Amini *et al.*, (2014) are:

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} g(x) \left\{-\log[G(x)]\right\}^{\alpha-1} [G(x)]^{\beta-1} \tag{2.28}$$

and

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} g(x) \left\{-\log[1 - G(x)]\right\}^{\alpha-1} [1 - G(x)]^{\beta-1} \tag{2.29}$$

where $\alpha > 0$ and $\beta > 0$ are additional shape parameters.

The statistical properties of this family of distributions were discussed including an application in Bayesian inference. Amini *et al.*, (2014) stressed that this family of distributions could be used to model more complex data sets. Unfortunately, this particular family of distributions has not been widely explored to generalise several known theoretical standard distributions and literature on it is very scanty.

2.2.10 Exponentiated T-X Family of Distributions

The Exponentiated T-X family of distributions was introduced by Alzaghal, Lee and Famoye (2013). It was formulated using the transformation:

$$F(x) = \int_0^{-\log\{1-G^a(x)\}} w(t)dt \qquad ; \qquad a > 0 \tag{2.30}$$

where $w(t)$ is the pdf of an arbitrary random variable T.

The corresponding pdf of the Exponentiated T-X family of distribution is:

$$f(x) = \frac{ag(x)G^{a-1}(x)}{1 - G^a(x)} r\left\{-\log[1 - G^a(x)]\right\} \qquad ; \qquad a > 0 \tag{2.31}$$

where a is a shape parameter, $G(x)$ and $g(x)$ are the cdf and pdf of the baseline distribution respectively.

This form of family of distributions can be used to generate some other families of distributions. For instance, generalised families like the Exponentiated Gamma-X family of distributions, Exponentiated Weibull-X family of distributions have been proposed by Alzaghal *et al.*, 2013. It was mentioned that it is possible for new families of discrete distributions to be generated by assuming the random variable X to be discrete. The T-X distribution was identified to be its special case. As an illustration, a compound distribution; Exponentiated Weibull-Exponential distribution (EWED) was developed. It has the Weibull distribution, Exponential distribution, Kumaraswamy Standard Exponential distribution, Standard Exponentiated Exponential distribution and the Type II Extreme Value distribution as its special cases.

Statistical properties of the Exponentiated Weibull-Exponential distribution which include the quantile function, moments, moment generating function, skewness and kurtosis. It was fitted to three real life data sets which relate to data on strengths of 1.5 cm glass fibers, data on depressive condition and data on repair times for an airborne communication transceiver. In all these applications, the Exponentiated Weibull-Exponential distribution provides better fits than the Beta Exponential distribution, Beta Frechet distribution, Beta Generalised Exponential distribution, Generalised Rayleigh distribution, Exponentiated Generalised Rayleigh distribution and Beta Generalised Rayleigh distribution.

Without doubts, the Exponentiated T-X family of distribution has been illustrated to be flexible but it has not been widely embraced by authors.

2.2.11 Logistic-G Family of Distributions

The Logistic-G family of distributions was introduced by Torabi and Montazari (2014). It was generated from the Logistic random variable and particularly via the transformation:

$$F(x) = \int_{0}^{\log\left[\frac{G(x)}{1-G(x)}\right]} w(t)dt \tag{2.32}$$

where $w(t)$ is the pdf of the Logistic distribution.

This concept has been used to derive the Logistic-Uniform distribution by Torabi and Montazari (2014). Estimation of parameters, Shannon entropy and simulation studies were examined. Its hazard rate function could exhibit concave-convex shape and J-bathtub. Also, an illustration was made to reveal the potentials of the Logistic-Uniform distribution over the Beta-Exponential distribution, Birnbaum-Saunders distribution, Beta Generalised-Exponential distribution, Beta-Normal, Beta Generalised half-Normal distribution, Beta-Laplace distribution, Gamma-Uniform distribution, Beta Modified Weibull distribution, Generalised Modified Weibull distribution, Beta Generalised Pareto distribution, Beta-Pareto distribution, Modified-Weibull distributions distribution and Beta-Weibull distribution.

The Logistic-G family of distribution seems flexible based on the results of Torabi and Montazari (2014) but it has not been rigorously explored with other authors. Also, literature on this form of generalisation is scanty.

2.2.12 Gamma-X Family of Distributions

Gamma-X family of distributions was defined by Alzaatreh, Famoye and Lee (2014). It was generated from the Gamma random variable. Particularly, its cdf is given as:

$$F(x) = \frac{\gamma\left\{\alpha, \frac{-\log[1-G(x)]}{\beta}\right\}}{\Gamma(\alpha)} \qquad ; \qquad x > 0 \tag{2.33}$$

where $\alpha > 0, \beta > 0$ are shape parameters and $\gamma(\alpha, x) = \int_0^x u^{\alpha-1}e^{-u}du$ is an incomplete gamma function.

The pdf of the Gamma-X family of distributions is:

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} g(x) \left\{-\log\left[1 - G(x)\right]\right\}^{\alpha-1} \left[1 - G(x)\right]^{\left(\frac{1}{\beta}\right)-1} \qquad ; \qquad x > 0 \tag{2.34}$$

This form of generalisation has been used to define the Gamma-Normal distribution and it was studied extensively in Alzaatreh *et al.*, (2014). Its shape was established, the hazard rate function, quantile function, Shannon entropy, moments and estimation of parameters were also derived. It was fitted to two real life data sets which relate to the breaking stress of carbon fibers of 50 mm in length and data set on the strengths of 1.5 cm glass fibres. The two-parameter Gamma-Normal distribution was reported to provide better fits than the four-parameter Gamma-Normal distribution, BirnbaumSaunders distribution and Beta BirnbaumSaunders distribution.

2.2.13 Logistic-X Family of Distributions

The Logistic-X family of distributions was developed by Tahir *et al.*, (2016). It was generated from the Logistic random variable. Its cdf was derived from the transformation:

$$F(x) = \int_0^{\log\{-\log[1-G(x)]\}} r(t)dt \tag{2.35}$$

where $r(t)$ is the pdf of the Logistic distribution and T is a random variable.

Hence, Tahir *et al.*, (2016) defined the cdf of the Logistic-X family of distributions as:

$$F(x) = \left\langle 1 + \{-\log[1 - G(x)]\}^{-\lambda} \right\rangle^{-1} \tag{2.36}$$

for $x > 0, \lambda > 0$

The pdf is:

$$f(x) = \frac{\lambda g(x)}{[1 - G(x)]} \{-\log[1 - G(x)]\}^{-(\lambda+1)} \left\langle 1 + \{-\log[1 - G(x)]\}^{-\lambda} \right\rangle^{-2} \tag{2.37}$$

for $x > 0, \lambda > 0$.

where λ is a shape parameter.

The Logistic Frechet distribution, Logistic Uniform distribution, Logistic Logistic distribution, Logistic Burr XII distribution, Logistic Weibull distribution and the Logistic Pareto distribution were also defined by Tahir *et al.*, (2016). General expressions for the moments, generating function, mean deviations, distribution of order statistics, Shannon entropy and estimation of parameters were also provided. In particular, the Logistic Frechet distribution was applied to two lifetime data sets which relate to data on survival times of guinea pigs and data on annual maximum rainfall measurements. It was reported that the Logistic Frechet distribution provides a better fit than the Marshal-Olkin Frehet distribution, Exponentiated Frechet distribution and the Frechet distribution.

2.2.14 Marshall Olkin Family of Distributions

The Marshall Olkin family of distributions was introduced by Marshall and Olkin (1997). It has only one additional parameter. The cdf of the Marshall Olkin family of distribution is:

$$F(x) = \frac{G(x)}{1 - \alpha\left[1 - G(x)\right]} \quad ; \quad x > 0, \alpha > 0 \tag{2.38}$$

Its pdf is:

$$f(x) = \frac{(1-\alpha)\, g(x)}{\left\{1 - \alpha\left[1 - G(x)\right]\right\}^2} \quad ; \quad x > 0, \alpha > 0 \tag{2.39}$$

where α is a shape parameter.

This form of generalisation has been used to derive the Marshall-Olkin Pareto type I distribution by Ghitany (2005) as a compound distribution with mixing exponential model. Marshall Olkin Gamma distribution was developed by Ristic, Jose and Ancy (2007) and Ghitany, Al-Awadhi and Alkhalfan (2007) also confirmed that the Marshall Olkin Lomax distribution is a compound distribution with mixing exponential model, the shape of its density and hazard rate function was established and an application to a censored data set was provided.

The Marshall Olkin Normal distribution (Garcia, Gomez-Deniz and Vazquez-Polo, 2010); Marshall Olkin Geometric distribution (Gomez-Deniz, 2010); Marshall Olkin Extended Weibull distribution (Cordeiro and Lemonte, 2013) and Marshall Olkin Frechet distribution (Krishna, Jose, Alice and Ristic, 2013) have also been developed.

Jose and Krishna (2011) derived the densities of the Marshall-Olkin Extended Uniform distribution and the reliability function, hazard function, mean, variance, quantiles, distribution of order statistics and estimation of parameters were discussed in details. Its possible application in autoregressive time series modelling was also discussed.

Gui (2013) derived the densities of the Marshall-Olkin Extended Log-logistic distribution and the expressions for its reliability function, hazard rate function, moments, quantile function and distribution of order statistics were provided. A minification process was constructed and its covariance structure was discussed. Although, an application to a real life data set was not provided.

Al-Saiari, Baharith and Mousa (2014) developed the densities of the Marshall-Olkin Extended Burr Type XII distribution, the Burr XII distribution and Marshall-Olkin Extended Lomax distribution have been identified as its special cases. Its shape was established, statistical properties like the quantile function has also been derived. The maximum likelihood estimation and Bayesian estimation of the Marshall-Olkin Extended Burr Type XII distribution has also been rigorously derived. An application to a data set which relates to data on electrical insulating was provided and the Marshall-Olkin Extended Burr Type XII distribution provides a better fit than the Burr XII distribution.

El-Nadi, Fatehy and Ahmed (2017) developed the densities of the Marshall-Olkin Exponential Pareto distribution, its moments and estimation of parameters were discussed and an application to a data set which relates to the remission times of bladder cancer patients confirmed that the Marshall-Olkin Exponential Pareto distribution provides a better fit than the Gamma distribution, Weibull distribution, Extended Weibull distribution and the Generalised Gamma distribution.

Also, Okorie, Akpanta and Ohakwe (2017) explored the Marshall-Olkin Generalised Erlang-truncated Exponential distribution, its asymptotic behaviour was investigated, statistical properties like the reliability function, hazard rate function, mean residual lifetime, quantile function, moments, moment generating function, Renyi entropy, distribution of order statistics and estimation of parameters were derived. A simulation study was conducted while drawing random samples of sizes $n = 50, 100, ..., 300$ from the Marshall-Olkin Generalised Erlang-truncated Exponential distribution and the stability of the parameters were investigated. An application to a data set which relates to the waiting time of bank customers confirmed that the Marshall-Olkin Generalised Erlang-truncated Exponential distribution provides a better fit than the Exponentiated Frechet distribution, Exponentiated Truncated Exponential distribution and Exponentiated Burr XII distribution, Another application to a rainfall data set establishes the same result.

2.2.15 Beta Marshall Olkin Family of Distributions

The Beta-Marshall-Olkin family of distributions was introduced by Alizadeh, Cordeiro, de Brito and Demetrio (2015a). It has three additional shape parameters. It was derived from the transformation:

$$F(x) = \int_0^{\frac{G(x)}{c+(1-c)G(x)}} r(t)dt \tag{2.40}$$

where $r(t)$ is the pdf of the Beta distribution with parameters a and b.

That is:

$$r(t) = \frac{1}{B(a,b)} t^{a-1} (1-t)^{b-1} \tag{2.41}$$

for $0 < t < 1$ and $a, b, c > 0$.

The cdf of the Beta-Marshall-Olkin family of distribution is:

$$F(x) = I_{\frac{G(x)}{c+(1-c)G(x)}} (a, b) \tag{2.42}$$

where $I_x (a, b) = \frac{1}{B(a,b)} \int_0^x t^{a-1} (1-t)^{b-1} dt$ is an incomplete beta function ratio.

The pdf of the Beta Marshall Olkin family of distributions is:

$$f(x) = \frac{c^b g(x)[G(x)]^{a-1} [1 - G(x)]^{b-1}}{B(a,b) [c + (1-c) G(x)]^{a+b}} \tag{2.43}$$

for $x > 0, a > 0, b > 0, c > 0$

where a, b and c are additional shape parameters.

General expression for the quantile power series, moments, generating function, mean deviations, entropies, distribution of order statistics and estimation of parameters have been provided in Alizadeh *et al.*, (2015a). As an illustration, densities for the Beta Marshall Olkin Normal distribution, Beta Marshall Olkin Weibull distribution and Beta Marshall Olkin Gamma distribution were also derived. An application to a data set which relates to data on failure times for a particular wind shield device revealed that the Beta Marshall Olkin Weibull distribution provides a better fit than the Weibull distribution, Exponentiated Weibull distribution, Beta Weibull distribution, Marshall-Olkin Weibull distribution, Exponentiated Marshall-Olkin Weibull distribution, Libby-Novic Beta Normal distribution and Kumaraswamy Weibull distribution.

This form of generalisation is relatively new and much work has not been done in that regard.

2.2.16 Kumaraswamy Marshall Olkin Family of Distributions

The Kumaraswamy Marshall-Olkin family of distributions was introduced by Alizadeh, Tahir, Cordeiro, Mansoor, Zubair and Hamedani (2015b). It involves three extra shape parameters. It was generated by first, making the Marshall Olkin family of distributions a baseline distribution and inserting it into the Kumaraswamy-G family of distributions. The cdf of the Kumaraswamy Marshall-Olkin family of distributions is:

$$F(x) = 1 - \left\langle 1 - \left\{ \frac{G(x)}{1 - c\left[1 - G(x)\right]} \right\}^{a} \right\rangle^{b} \tag{2.44}$$

for $x > 0, a > 0, b > 0, c > 0$

Its pdf is:

$$f(x) = \frac{ab\,(1-c)\,g(x)[G(x)]^{a-1}}{\{1-c\,[1-G(x)]\}^{a+1}} \left\langle 1 - \left\{ \frac{G(x)}{1-c\,[1-G(x)]} \right\}^{a} \right\rangle^{b} \tag{2.45}$$

for $x > 0, a > 0, b > 0, c > 0$

where a, b and c are additional shape parameters.

This form of generalisation has the Kumaraswamy-G family of distributions, Marshall-Olkin family of distribution, Exponentiated Marshall-Olkin family of distribution, Proportional reversed hazard rate models and Proportional hazard rate model as special cases. As illustrations, the Kumaraswamy Marshall-Olkin Exponential distribution, Kumaraswamy Marshall-Olkin Lomax distribution, Kumaraswamy Marshall-Olkin Weibull distribution and Kumaraswamy Marshall-Olkin Frechet distribution have been derived by Alizadeh *et al.*, (2015b). An application to a data set which relates to the survival times of patients given chemotherapy treatment alone was provided and the Kumaraswamy Marshall-Olkin Weibull distribution performes better than the Beta Weibull distribution, Kumaraswamy Weibull distribution, Exponentiated Weibull distribution, Marshall-Olkin Extended Weibull distribution and the Weibull distribution. Meanwhile, another application to a data set relating to the strength of single carbon fibers did not produce similar results when Kumaraswamy Marshall-Olkin Frechet distribution was used.

2.2.17 Kumaraswamy Transmuted-G Family of Distributions

The Kumaraswamy Transmuted-G family of distributions was defined by Afify, Cordeiro, Yousof, Alzaatreh and Nofal (2016). It was generated by combining the works of Cordeiro and de Castro (2011) and that of Shaw and Buckley (2007). In clearer terms; the Kumaraswamy generalised family of distributions and the Transmuted family of distributions were combined. This form of generalisation has three additional parameters. The cdf of the Kumaraswamy Transmuted-G family of distributions is:

$$F(x) = 1 - \left\{1 - \left[(1+\lambda)\,G(x) - \lambda[G(x)]^2\right]^{a}\right\}^{b} \tag{2.46}$$

for $a > 0, b > 0, |\lambda| \leq 1$

Its pdf is:

$$f(x) = abg(x)\left[1+\lambda-2\lambda G(x)\right]\left\{G(x)\left[1+\lambda-\lambda G(x)\right]\right\}^{a-1} \times \left\{1-\left[(1+\lambda)\,G(x)-\lambda[G(x)]^{2}\right]^{a}\right\}^{b-1} \quad (2.47)$$

for $a > 0, b > 0, |\lambda| \leq 1$

where a and b are shape parameters and λ is the transmuted parameter.

This form of generalisation has the Transmuted family of distribution and the Kumaraswamy-G family of distribution as special cases. As illustrations, the Kumaraswamy Transmuted Exponential distribution, Kumaraswamy Transmuted Power distribution, Kumaraswamy Transmuted Log-logistic distribution and the Kumaraswamy Transmuted Burr X distribution have been derived by Afify *et al.*, (2016). The Kumaraswamy Transmuted-G family of distribution is relatively new and it has not been been widely used by authors.

2.3 Existing Generalisations of Inverse Exponential Distribution

Some works that have been done which involve the generalisation of the Inverse Exponential distribution are discussed next. To be precise, the Generalised Inverse Exponential distribution and the Beta Inverse Exponential distribution are available in the literature and are discussed in some details as they relate to this study.

2.3.1 The Generalised Inverse Exponential (GIE) Distribution

Abouammoh and Alshingiti (2009) were the first to generalise the Inverse Exponential distribution. An additional parameter was introduced to the Inverse Exponential distribution and the resulting distribution was named Generalised Inverse Exponential distribution. It can also be called Generalised Inverted Exponential distribution and it has two parameters.

The cdf of the GIE distribution is:

$$F(x) = 1 - \left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha} \qquad ; \qquad x > 0, \alpha > 0, \theta > 0 \tag{2.48}$$

Its pdf is:

$$f(x) = \frac{\alpha\theta}{x^2} e^{-\frac{\theta}{x}} \left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha-1} \qquad ; \qquad x > 0, \alpha > 0, \theta > 0 \tag{2.49}$$

where α is the shape parameter and θ is the scale parameter.

Its survival function and hazard function are given as:

$$S(x) = \left(1 - e^{-\frac{\theta}{x}}\right)^{\alpha} \qquad ; \qquad x > 0, \alpha > 0, \theta > 0 \tag{2.50}$$

and

$$h(x) = \frac{\alpha\theta}{x^2} \left(e^{-\frac{\theta}{x}} - 1\right) \qquad ; \qquad x > 0, \alpha > 0, \theta > 0 \tag{2.51}$$

respectively.

It can be noted that the GIE distribution reduces to give the Inverse Exponential distribution when $\alpha = 1$.

Statistical properties of the GIE distribution were studied as well as estimation of parameters. Its application was demonstrated using a real life data set relating to tests on endurance of deep groove ball bearings. It was discovered that the GIE distribution performs better than the Inverse Exponential distribution using the Likelihood Ratio Test and the Kolmogorov Smirnov Statistic as selection criteria. It was also noted that the GIE distribution provides a better fit than the Gamma distribution.

The GIE distribution is positively skewed, unimodal, and the shape of the hazard function could be decreasing or increasing. According to Dey and Pradhan (2014), the GIE distribution can be used in accelerated life testing, horse racing, queue theory, modelling wind speeds, e.t.c. Some limitations of the work of Abouammoh and Alshingiti (2009) are noticed; the work did not provide a simulation study to investigate the behaviour of the parameters of the GIE distribution. Also, the ability of the GIE distribution in modelling real life data sets is limited; this is shown and catered for in the result section of this study.

2.3.2 The Beta Inverted Exponential (BIE) Distribution

The Beta Inverted Exponential distribution which can also be called Beta Inverse Exponential distribution was introduced recently by Singh and Goel (2015). It was derived using the Beta Generalised family of distributions that was proposed by Eugene *et al.*, (2002).

The Inverse Exponential distribution was made to be the baseline distribution and the resulting distribution has three parameters all together; two shape parameters and a scale parameter. The cdf of the BIE distribution is given as:

$$F(x) = \frac{1}{B(a,b)} \int_0^{e^{-\frac{\theta}{x}}} w^{a-1} (1-w)^{b-1} dw \tag{2.52}$$

for $x > 0, a > 0, b > 0, \theta > 0$

Its pdf is:

$$f(x) = \frac{1}{B(a,b)} \frac{\theta}{x^2} e^{-\frac{a\theta}{x}} \left(1 - e^{-\frac{\theta}{x}}\right)^{b-1} \tag{2.53}$$

for $x > 0, a > 0, b > 0, \theta > 0$

where θ is the scale parameter, a and b are shape parameters.

The Generalised Inverse Exponential (GIE) distribution can be obtained from the BIE distribution when parameter $a = 1$.

Also, the Inverse Exponential (IE) distribution can be derived from the BIE distribution when parameters $a = b = 1$.

Several statistical properties of the BIE distribution were derived. The shape of the hazard function for the BIE distribution could be non-monotonic and inverted bathtub. It was noted that the distribution can be used to model lifetime data sets.

A simulation study was conducted using R software by setting the values of the parameters at; $\theta = 3, a = 0.3, b = 2$ and $\theta = 5, a = 0.5, b = 4$. Hence, two different data sets were obtained, the sample sizes considered are $n = 30, 50, 80$ and 100. The purpose was to compare the performances of the maximum likelihood estimates (MLE) with that of the Bayes estimates.

The BIE distribution was applied to two real data sets relating to the fatigue time of aluminum coupons and the breaking stress of carbon fibers. Its performance was compared with that of Generalised Inverse Exponential distribution, Inverse Exponential distribution, Inverse Rayleigh distribution, Beta Weibull distribution and Beta Exponential distribution. The performance was judged based on the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) values posed by each of the distribution.

Considering the first lifetime data set that was used for the analysis, the GIE distribution performed better than the BIE distribution including the remaining competing distributions based on the value of the AICs and BICs.

For the second lifetime data set, the Beta Exponential distribution performed better than the BIE distribution including all the other competing distributions based on the value of AICs and BICs.

There are some shortcomings that could be noticed in the study of Singh and Goel (2015). First, dealing with the BIE distribution could lead to some mathematical complexities because the cdf of the BIE distribution is not tractable, it involves a special function.

Secondly, the BIE distribution may not also be flexible; this is because, out of the two data sets used by Singh and Goel (2015), the BIE distribution did not perform better than the other competing distributions based on the AIC and BIC values.

2.4 Summary

In this chapter, the various generalised families of distributions have been examined in details. Various authors that have used these different families of distributions have also been identified including the distributions derived. More importantly, two different works which also involved the generalisation of the Inverse Exponential distribution; the GIE distribution and the BIE distribution have been identified including their gaps.

Discussed in the next chapter are the methods used to achieve the stated objectives in this study.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

The various methods that were used to actualise the stated objectives for this study are discussed and outlined in this chapter. However, R software was adopted in performing all the analyses in this study.

3.2 Statistical Table for the Inverse Exponential Distribution

The cdf and pdf of the Inverse Exponential distribution are given as:

$$G(x) = exp\left(-\frac{\theta}{x}\right) \qquad ; \qquad x > 0, \theta > 0 \tag{3.1}$$

and

$$g(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right) \qquad ; \qquad x > 0, \theta > 0 \tag{3.2}$$

respectively.

where θ is the scale parameter

To actualise objective 1, the following were performed using R software:

i An algorithm to represent the Inverse Exponential distribution was written. The details about this algorithm is available in Appendix A.

ii Values for the Inverse Exponential distribution for $x = 1, 2, 3, ..., 25$ and for selected values of $\theta = 0.5$ to $\theta = 10$ were generated. These parameter values were selected arbitrarily and for brevity purpose.

Discussed next is how the Kumaraswamy Inverse Exponential disribution was derived.

3.3 Deriving The Kumaraswamy Inverse Exponential Distribution

Let X denote a non-negative random variable from an arbitrary distribution whose pdf and cdf are defined by $g(x)$ and $G(x)$ respectively, following Cordeiro and de Castro (2011), the cdf of the Kumaraswamy-G distribution is given as:

$$F(x) = 1 - \{1 - [G(x)]^a\}^b \tag{3.3}$$

for $x > 0, a > 0, b > 0$

Now, to derive the expression for the cdf of the Kumaraswamy Inverse Exponential distribution, the works of Keller and Kamath (1982) and that of Cordeiro and de Castro (2011) were combined. This was achieved by inserting the expression in Equation (3.1) as the cdf of the parent distribution into the expression in Equation (3.3), the resulting compound distribution gives the cdf of the Kumaraswamy Inverse Exponential distribution.

Also, since the pdf of the Kumaraswamy-G distribution is given as:

$$f(x) = abg(x)[G(x)]^{a-1}\{1 - [G(x)]^a\}^{b-1} \tag{3.4}$$

for $x > 0, a > 0, b > 0$

where a and b are shape parameters.

Then, the expressions in Equations (3.1) and (3.2) were inserted into the expression in Equation (3.4) to give the pdf of the Kumaraswamy Inverse Exponential distribution. Discussed next is how the Transmuted Inverse Exponential disribution was derived.

3.4 Deriving The Transmuted Inverse Exponential Distribution

Let X denote a non-negative random variable from an arbitrary distribution whose pdf and cdf are defined by $g(x)$ and $G(x)$ respectively. According to Shaw and Buckley (2007), the cdf of the Transmuted-G distribution is given as:

$$F(x) = (1 + \lambda)\, G(x) - \lambda[G(x)]^2 \tag{3.5}$$

for $x > 0$ and $|\lambda| \leq 1$

To derive the cdf of the Transmuted Inverse Exponential distribution, the works of Keller and Kamath (1982) and that of Shaw and Buckley (2007) were combined. This was achieved by inserting Equation (3.1) as the cdf of the parent distribution into the expression in Equation (3.5) thus resulting into the cdf of the Transmuted Inverse Exponential distribution.

As given by Shaw and Buckley (2007), the corresponding pdf of the expression in Equation (3.5) is:

$$f(x) = g(x)\, [1 + \lambda - 2\lambda G(x)] \tag{3.6}$$

for $x > 0$ and $|\lambda| \leq 1$

where λ is the transmuted parameter.

Therefore, the pdf of the Transmuted Inverse Exponential was obtained by substituting Equations (3.1) and (3.2) into Equation (3.6). Discussed next is how the Exponentiated Generalised Inverse Exponential disribution was derived.

3.5 Deriving The Exponentiated Generalised Inverse Exponential Distribution

Let X denote a non-negative random variable from an arbitrary distribution whose pdf and cdf were defined by $g(x)$ and $G(x)$ respectively. Cordeiro *et al.*, (2013) gave the cdf of the Exponentiated Generalised-G distribution as:

$$F(x) = \{1 - [1 - G(x)]^a\}^b \tag{3.7}$$

for $x > 0, a > 0, b > 0$

Now, by combining the works of Keller and Kamath (1982) and that of Cordeiro *et al.*, (2013), the cdf of the Exponentiated Generalised Inverse Exponential (EGIE) distribution was derived. In particular, the cdf of the EGIE distribution was derived by inserting Equation (3.1) which is the cdf of the parent distribution into Equation (3.7).

The pdf corresponding to the expression in Equation (3.7) is:

$$f(x) = abg(x)\left[1 - G(x)\right]^{a-1}\left\{1 - \left[1 - G(x)\right]^{a}\right\}^{b-1} \tag{3.8}$$

for $x > 0, a > 0, b > 0$

where a and b are the shape parameters.

Therefore, the pdf of the EGIE distribution was obtained by substituting Equations (3.1) and (3.2) into Equation (3.8) and the resulting expression simplified. Discussed next is how the Weibull Inverse Exponential distribution was derived.

3.6 Deriving The Weibull Inverse Exponential Distribution

Let X denote a non-negative random variable from an arbitrary distribution whose pdf and cdf were defined by $g(x)$ and $G(x)$ respectively. Bourguinon *et al.*, (2014) gave the cdf of the Weibull-G distribution as:

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1 - G(x)}\right]^{\beta}\right\} \tag{3.9}$$

for $x > 0, \alpha > 0, \beta > 0$

When the works of Keller and Kamath (1982) and that of Bourguinon *et al.*, (2014) are combined, then the cdf and pdf of the Weibull-Inverse Exponential (WIE) distribution were obtained. Particularly, the cdf of the WIE distribution was derived by inserting Equation (3.1) into Equation (3.9).

The pdf corresponding to the expression in Equation (3.9) is:

$$f(x) = abg(x)\frac{\left[G(x)\right]^{\beta-1}}{\left[1 - G(x)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x)}{1 - G(x)}\right]^{\beta}\right\} \tag{3.10}$$

for $x > 0, \alpha > 0, \beta > 0$

where α and β are shape parameters.

Therefore, the pdf of the WIE distribution was derived by inserting Equations (3.1) and (3.2) into Equation (3.10) after which the resulting expression was simplified.

However, for each of the derived compound distribution:

i Their shape(s) are established with the aid of graphs. In particular, the algorithm developed for plotting the graph of the KIE distribution is available in Appendix B.

ii Their various sub-models are identified.

iii Their basic statistical properties are derived including estimation of model parameters.

iv Their application to real data sets are provided and an assessment of their superiority over related sub-models are provided.

Remark: The discussions in sub-sections 3.3 up to 3.6 cater for objective 2.

3.7 Deriving the Various Statistical Properties of the Proposed Generalised Distributions

To achieve objective 3, the statistical properties that were considered in this study are the asymptotic behaviour, survival function, hazard function, quantile function and distribution of order statistics.

3.7.1 Asymptotic Behaviour

The asymptotic behaviour can sometimes be called asymptotic properties or limiting behaviour. It involves studying the behaviour of the distributions at $x = 0$ and as $x \to \infty$. In this case, what was considered is:

$$\lim_{x \to \delta} F(x)$$

as $\delta \to \infty$. For each of the cases considered, it is expected that $\lim_{x \to \delta} F(x)$ will equal to or approach one.

3.7.2 Survival Function

The survival function can also be called the reliability function or survivor function. Generally, its mathematical representation is given as:

$$S(x) = Pr\,(X > x)$$

$$= 1 - Pr\,(X \leq x)$$

$$S(x) = 1 - F(x) \tag{3.11}$$

In this context, $F(x)$ is the cdf of any of the new compound distributions.

3.7.3 Hazard Function

In general, the hazard function is given as:

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \tag{3.12}$$

The hazard function for each of the new compound distributions were derived using the relation in Equation (3.12). Also, the shape(s) of the hazard function for each of the new compound distributions are established with the aid of graphs using R software.

3.7.4 Quantile Function

The quantile function is another way of describing a probability distribution. It can also be called the inverse cdf. It can be used to generate random samples for probability distributions and thereby can serve as an alternative to the pdf. In general, it is given as:

$$Q(u) = F^{-1}(u) \tag{3.13}$$

where U follows a Uniform distribution. That is, $U \sim Uniform(0, 1)$.

3.7.5 Distribution of Order Statistics

Order Statistics is an interesting topic in Statistics, suppose that $X_1, X_2, ..., X_n$ are independently and identically distributed random variables, each having a pdf and cdf of $f(x)$ and $F(x)$ respectively. If $Y_1 < Y_2 < .... < Y_n$ represent the $X_i's$ arranged in ascending order, then $Y_1 = Min(X_i)$ and it can be said that $Y_n = Max(X_i)$. In that sense, the $Y_k's$ are referred to as the order statistics of the $Y_i's$.

The distribution of k^{th} order statistics is given as:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) \left[F(x)\right]^{k-1} \left[1 - F(x)\right]^{n-k} \tag{3.14}$$

In this study:

$f(x)$ is the pdf of the generalised distribution under study

$F(x)$ is the cdf of the generalised distribution under study

Remarks:

i If $k = 1$ in Equation (3.14), the expression reduces to give the distribution of minimum order statistics.

ii If $k = n$ in Equation (3.14), the expression reduces to give the distribution of maximum order statistics.

3.8 Estimation of Parameters

To estimate the various unknown parameters resulting from the proposed generalised distributions, the method of Maximum Likelihood Estimation (MLE) was used. Though, there are other methods that could be used for parameter estimation, the commonly used methods include:

i Maximum Likelihood Estimation (MLE).

ii Bayesian Estimation.

iii Method of Moments (MOM).

iv Least Square Estimation.

This study has a special bias for the MLE and the reasons for this choice are stated as follows:

i The algorithms for obtaining the maximum likelihood estimates for majority of the widely known distributions are readily available and installed in several and major statistical software. This in turn makes work easier and reduces the computational complexities of the MLE.

ii The method of MLE has lower variance than the other methods.

iii It provides a consistent approach to parameter estimation problems. That is, it can be developed for a large variety of estimation problems.

iv As sample size increases, maximum likelihood estimates become minimum variance unbiased estimators.

There are also some disadvantages for using the method of MLE, these disadvantages include:

i It involves computational efforts and this method might be slow except for the use of statistical software.

ii The equations must be systematically worked out for every given distribution and estimation under study.

iii The method can be biased if small samples are used.

The use of the MLE involves getting the joint pdf for all the observations. In particular, for independent and identically distributed random samples, a maximum likelihood estimator of a parameter θ for a pdf $f(x;\theta)$ is an estimator that maximises the likelihood function $L(x_1, x_2, ..., x_n; \theta)$ as a function of θ (Hoel, 1954).

$$L(x_1, x_2, ..., x_n; \theta) = f(x_1; \theta) \times f(x_2; \theta) \times ...... \times f(x_n; \theta) \tag{3.15}$$

In general, the expression in Equation (3.15) can be written as:

$$L(x_1, x_2, ..., x_n; \theta) = \prod_{i=1}^{n} [f(x_i; \theta)] \tag{3.16}$$

where θ represents the parameter to be estimated.

The procedure of MLE to estimate the unknown parameters is presented as follows:

i The likelihood function is obtained.

ii The logarithm of the likelihood function is further obtained to give the log-likelihood function.

iii The partial derivatives of the log-likelihood function are obtained with respect to the various parameters under study.

iv The resulting derivatives are equated to zero and solved simultaneously.

Remarks: For compound distributions, the maximum likelihood estimates may be difficult to obtain analytically. Meanwhile, they can be obtained numerically with the aid of statistical software.

3.9 Criteria for Selecting the Distribution with the 'Best Fit'

To achieve objective 4, all the new compound distributions derived in this study were applied to real life data sets. Details about the data sets are given in the next subsection.

The aim of the real life application is to select the best distribution when comparisons are made with respect to related existing distributions. To achieve this, the following criteria were used; Likelihood Ratio Test (LRT) and Akaike Information Criterion (AIC).

The hypothesis tested in any of the cases is of the form:

$$H_0 : \theta = \theta_0$$

Versus

$$H_1 : \theta \neq \theta_0$$

The LRT is based on the likelihood ratio given as:

$$\lambda = \frac{L(\theta_0)}{L(\hat{\theta})} \tag{3.17}$$

The procedure involves calculating the maximum likelihood estimates from the data set based on an assumed distribution. This calculation is repeated for other candidate distributions. Then, the distribution with the largest likelihood value was selected as the best.

The AIC is computed as:

$$AIC = -2 \log L + 2k \tag{3.18}$$

where $\log L$ is the maximised value of the log-likelihood function under the distribution considered and k is the number of parameters.

The distribution with the smallest AIC or the highest log-likelihood value corresponds to the best distribution among the distributions under study.

Remark: AIC is used instead of some other criteria because it is asymptotically efficient.

3.10 Simulation Study

A data set of size $m = 1,000$ was simulated from the Weibull distribution using the command 'population=rweibull(1000,2,3)' and random samples of sizes $n = 50$, 100, 150, 200, 500 and 800 were drawn from it using the command 'x=sample(population,50)', 'x=sample(population,100)' and so on. Details about the algorithm used is available in Appendix E. A newly derived compound distribution was fitted to the data sets and the maximum likelihood estimates (MLEs) of the parameters were computed using the maxlik function in R software.

Remark: Weibull distribution was selected for the simulation because of its wide usage and acceptability in modelling real life phenomena in engineering and medicine. In addition, Weibull distribution is the most important distribution for problems in reliability (Bourguignon *et al.*, 2014).

The aim of the simulation study is to know if:

i The difference(s) between the average estimates and true values would be small or not.

ii The MLEs would converge to the true value in all cases when as the sample size increases.

iii The standard errors of the MLEs would decrease as the sample size increases.

The accuracy of the approximation of the standard error of the MLEs was assessed and the criteria for evaluating the performance of the estimators in this study are;

i Bias: $Bias = E(\hat{\theta}) - \theta$

ii Absolute Bias: $abs\left(E(\hat{\theta}) - \theta\right) = |E(\hat{\theta}) - \theta|$

iii Mean Square Error (MSE): $MSE(\hat{\theta}) = \frac{1}{n}E(\hat{\theta} - \theta)^2$

3.11 Deriving The Exponential-G Family of Distributions

To achieve objective 5, the following transformation was considered:

$$\Phi(x) = \int_0^{\frac{\Xi(x)}{1-\Xi(x)}} \alpha e^{-\alpha t} dt \qquad (3.19)$$

Where $\Xi(x)$ is the cdf of any arbitrary parent distribution.

The resulting expression serves as the cdf of the Exponential-G family of distributions. The result can also be obtained directly from the work of Bourguignon *et al.*, (2014) when $\beta = 1$. However, Bourguignon *et al.*, (2014) did not explore this particular special case. The idea in this study is to explore a case where the family of distribution will have only one additional parameter and yet be tractable.

To this end, the following were performed:

i Some basic general statistical properties for the Exponential-G family of distribution were derived and established.

ii Some compound distributions were defined using the Exponential-G family of distributions.

iii The flexibility of the Exponential-G family of distributions was assessed using real life data sets.

3.12 Data Collection

This study is a quantitative research and the data sets used for further analysis in this study were obtained as secondary data and particularly, they were extracted from textbooks and referred journals. The data sets relate to the fields of engineering (DATA I, IV, IX and X), finance (DATA VII), medicine (DATA II, III, V, VI, XII and XII) and hydrology (DATA VIII). The data sets used are presented as follows:

DATA I: The first data set represents the times of failures and running times for sample of devices from an eld-tracking study of a larger system. The data set has been previously studied by Meeker and Escobar (1988); Merovci and Elbatal (2015b). In particular, it was used to investigate the superiority of the Weibull Rayleigh distribution over the Beta Weibull distribution, Exponentiated Weibull distribution and Weibull distribution. The data set has thirty (30) observations and they are as follows:

2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66

DATA II: The second data set represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The data set was given by Gross and Clark (1975), it has been used recently by Shanker, Fesshaye and Selvaraj (2015) to assess the flexibility of Exponential distribution and Lindley distribution. The data set consists of twenty (20) observations and it is as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2

DATA III: The third data set represents the death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile. The data set has been previously used by Sickle-Santanello, Farrar and Keyhani-Rofagha (1988); Klein and Moeschberger (2003); Oguntunde and Adejumo (2015); Oguntunde *et al.*, (2016). Apart from its application in discriminating between the Generalised Inverse Generalised Exponential distribution, Inverse Generalised Exponential distribution and the Inverse Exponential distribution, it has also been used to assess the superiority of the Exponentiated Generalised Exponential distribution over the Exponentiated Exponential distribution, Generalised Exponential distribution and Exponential distribution. The data set consists of fifty-two (52) observations out of which twenty-one (21) are censored observations. The data set is as follows:

1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61*, 65, 67, 70, 72, 73, 74*, 77, 79*, 80*, 81*, 87*, 87*, 88*, 89*, 91, 93*, 96, 97*, 100, 101*, 104, 104*, 108*, 109*, 120*, 131*, 150*, 157, 167, 231*, 240*, 400*

NOTE: * denote censored observations

DATA IV: The fourth data set represents the failure times of the air conditioning system of an airplane. The data set was given by Linhart and Zucchini (1986) and it has also been used by Shanker *et al.*, (2015). In particular, it has previously been used to discriminate between the Exponential distribution and the Lindley distribution. It has thirty (30) observations as follows:

23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95

DATA V: The fifth data set represents the survival times (in days) of seventy-two (72) guinea pigs infected with virulent tubercle bacilli. It has been previously used by Bjerkedal (1960); Shanker *et al.*, (2015). It was used to select the best distribution between the Exponential distribution and the Lindley distribution. The data set is given as follows:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376

DATA VI: The sixth data set represents the survival times of a group of patients suffering from Head and Neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (RT+CT). The data set has been previously used by Efron (1988) and Shanker *et al.*, (2015). It has been successfully used to assess the superiority of the Exponential distribution over the Lindley distribution. It has forty-four (44) observations and they are as follows:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776

DATA VII: The seventh data set represents the waiting time (mins) of one hundred (100) bank customers before service is being rendered. It has previously been used by Ghitany, Atieh and Nadarajah (2008). In particular, the Lindley distribution and Exponential distribution have been previously fitted to the data. The data set is as follows:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2,, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5

DATA VIII: The eight data set represents the vinyl chloride data (in mg/l) that was obtained from clean upgradient monitoring wells. It has been previously used by Bhaumik, Kapur and Gibbons (2009). It was particularly used to test the parameters of a Gamma distribution. The data set has thirty-four (34) observations and they are as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2

DATA IX: The ninth data set represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90 percent stress level until all had failed. The data set has been previously studied by Barlow, Toland and Freeman (1984); Andrews and Herzberg (1985); Abdul-Moniem and Seham (2015). Among other applications, it has been used to assess the superiority of the Transmuted Gompertz distribution over the Gompertz distribution. It has seventy-six (76) observations and they are as follows:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960

DATA X: The tenth data set represents the number of million revolutions before failure for each of twenty-three (23) deep groove ball bearings in the life tests. The data set was given by Lawless (1982) and it has also been used by Shanker *et al.*, (2015). It has been used to select the best distribution between the Exponential distribution and the Lindley distribution. The observations are as follows:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

DATA XI: The eleventh data set was given by Lee and Wang (2003) and it represents the remission times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients. Its application in survival analysis has been identified and the data set is as follows:

0.08, 2.09, 3.48,4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69.

DATA XII: The twelveth data set was given by Lee (1992) and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. It has also been applied by Ramos *et al.*, (2013). It has been used to discriminate between the Kumaraswamy Log-Logistic distribution, Beta Log-Logistic distribution, Exponentiated Weibull distribution and Exponentiated Log-Logistic distribution. The data set is as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0

3.13 Summary

In this chapter, the various methods used in this study have been provided including the hypothesis and the methods for the simulation study. These methods were discussed and presented with respect to the stated objectives.

Twelve secondary data sets have also been presented, the data sets relate to the field of engineering, finance, medicine and hydrology. These data sets as well as simulated data were used for the analyses in this study as shown in the next chapter. Meanwhile, the software adopted for data analysis in this study is R software, the performance of the compound distributions with respect to their related competing distributions are judged based on their log-likelihood and AIC values.

The next chapter presents the main results of this study as they relate to the stated objectives.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Introduction

The results obtained in this study are presented in this chapter; it includes statistical tables, derivations of the proposed compound distributions, derivations of the statistical properties of these compound distributions, plots for the pdf, cdf, survival function and hazard function for the newly derived compound distributions, analysis of the data sets collected and simulation study. Some algorithm for the execution of the analyses in R software are provided in Appendix A, B, C, D and E.

4.2 Statistical Table for the Inverse Exponential Distribution

In this study, only the statistical table for the pdf of the IE distribution is provided. The pdf of the Inverse Exponential (IE) distribution is as given in Equation (3.2). For brevity purpose, the parameter values considered for generating the statistical table ranges from $\theta = 0.5$ to $\theta = 10$ and the value of the variable considered is $x = 1, 2, 3, ..., 25$. The result is given in Tables 4.1 to 4.3 and the algorithm used is available in Appendix A:

Table 4.1: Statistical Table for the PDF of Inverse Exponential Distribution (for $\theta = 0.5$ to $\theta = 3.5$)

x	$\theta = 0.5$	$\theta = 1.0$	$\theta = 1.5$	$\theta = 2.0$	$\theta = 2.5$	$\theta = 3.0$	$\theta = 3.5$
1	0.30327	0.36788	0.33469	0.27067	0.20521	0.14936	0.10569
2	0.09735	0.15163	0.17714	0.18394	0.17907	0.16735	0.15205
3	0.04703	0.07961	0.10109	0.11409	0.12072	0.12263	0.12110
4	0.02758	0.04868	0.06443	0.07582	0.08363	0.08857	0.09119
5	0.01809	0.03275	0.04445	0.05363	0.06065	0.06586	0.06952
6	0.01278	0.02351	0.03245	0.03981	0.04578	0.05054	0.05425
7	0.00950	0.017691	0.02471	0.03067	0.03569	0.03988	0.04332
8	0.00734	0.01379	0.01943	0.02434	0.02858	0.03222	0.03531
9	0.00584	0.01105	0.01568	0.01977	0.02338	0.02654	0.02929
10	0.00476	0.00905	0.01291	0.01637	0.01947	0.02222	0.02466
11	0.00395	0.00755	0.01082	0.01378	0.01646	0.01888	0.02104
12	0.00333	0.00639	0.00919	0.01175	0.01409	0.01623	0.01816
13	0.00285	0.00548	0.00791	0.01015	0.01220	0.01409	0.01582
14	0.00246	0.00475	0.00688	0.00885	0.01067	0.01235	0.01391
15	0.00215	0.00416	0.00603	0.00778	0.00941	0.01092	0.01232
16	0.00189	0.00367	0.00534	0.00689	0.00835	0.00972	0.01099
17	0.00168	0.00326	0.00475	0.00615	0.00747	0.00870	0.00986
18	0.00150	0.00292	0.00426	0.00552	0.00672	0.00784	0.00889
19	0.00135	0.00263	0.00384	0.00499	0.00607	0.00709	0.00806
20	0.00122	0.00238	0.00348	0.00452	0.00552	0.00646	0.00735
21	0.00111	0.00216	0.00317	0.00412	0.00503	0.00589	0.00672
22	0.00101	0.00197	0.00289	0.00377	0.00461	0.00541	0.00617
23	0.00092	0.00181	0.00266	0.00347	0.00424	0.00498	0.00568
24	0.00085	0.00167	0.00245	0.00319	0.00391	0.00459	0.00525
25	0.00078	0.00154	0.00226	0.00295	0.00362	0.00426	0.00487

Table 4.2: Statistical Table for the PDF of Inverse Exponential Distribution (for $\theta = 4.0$ to $\theta = 7.0$)

x	$\theta = 4.0$	$\theta = 4.5$	$\theta = 5.0$	$\theta = 5.5$	$\theta = 6.0$	$\theta = 6.5$	$\theta = 7.0$
1	0.07326	0.04999	0.03369	0.02248	0.01487	0.00977	0.00638
2	0.13534	0.11857	0.10261	0.08790	0.07468	0.06301	0.05285
3	0.11715	0.11157	0.10493	0.09770	0.09022	0.08274	0.07542
4	0.09197	0.09131	0.08953	0.08691	0.08367	0.07999	0.07603
5	0.07189	0.07318	0.07357	0.07323	0.07229	0.07086	0.06905
6	0.05705	0.05905	0.06036	0.06109	0.06131	0.06111	0.06055
7	0.04609	0.04829	0.04995	0.05116	0.05196	0.05241	0.05255
8	0.03791	0.04006	0.04182	0.04321	0.04428	0.04507	0.04559
9	0.03166	0.03369	0.03542	0.03685	0.03803	0.03897	0.03970
10	0.02681	0.02869	0.03033	0.03173	0.03293	0.03393	0.03476
11	0.02298	0.02470	0.02622	0.02757	0.02874	0.02975	0.03062
12	0.01990	0.02147	0.02289	0.02415	0.02527	0.02626	0.02713
13	0.01739	0.01884	0.02014	0.02132	0.02238	0.02333	0.02417
14	0.01534	0.01664	0.01785	0.01894	0.01994	0.02085	0.02166
15	0.01362	0.01482	0.01592	0.01694	0.01788	0.01873	0.01951
16	0.01217	0.01327	0.01429	0.01523	0.01611	0.01691	0.01765
17	0.01094	0.01195	0.01289	0.01377	0.01459	0.01534	0.01605
18	0.00989	0.01082	0.01169	0.01251	0.01327	0.01398	0.01464
19	0.00898	0.00984	0.01065	0.01141	0.01212	0.01279	0.01341
20	0.00819	0.00898	0.00974	0.01044	0.01111	0.01174	0.01233
21	0.00749	0.00824	0.00894	0.00959	0.01022	0.01082	0.01137
22	0.00689	0.00758	0.00823	0.00885	0.00944	0.00999	0.01052
23	0.00635	0.00699	0.00761	0.00819	0.00874	0.00926	0.00976
24	0.00588	0.00648	0.00705	0.00759	0.00811	0.00861	0.00908
25	0.00545	0.00601	0.00655	0.00706	0.00755	0.00802	0.00846

Table 4.3: Statistical Table for the PDF of Inverse Exponential Distribution (for $\theta = 7.5$ to $\theta = 10.0$)

x	$\theta = 7.5$	$\theta = 8.0$	$\theta = 8.5$	$\theta = 9.0$	$\theta = 9.5$	$\theta = 10.0$
1	0.00415	0.00268	0.00173	0.00111	0.00071	0.00045
2	0.04409	0.03663	0.03031	0.02499	0.02055	0.01684
3	0.06840	0.06176	0.05555	0.04979	0.04449	0.03964
4	0.07189	0.06767	0.06345	0.05929	0.05523	0.05130
5	0.06694	0.06461	0.06211	0.05951	0.05684	0.05413
6	0.05969	0.05858	0.05726	0.05578	0.05417	0.05246
7	0.05243	0.05207	0.05151	0.05078	0.04990	0.04891
8	0.04589	0.04598	0.04589	0.04565	0.04527	0.04477
9	0.04024	0.04060	0.04081	0.04088	0.04081	0.04064
10	0.03543	0.03595	0.03633	0.03659	0.03674	0.03679
11	0.03134	0.03195	0.03244	0.03282	0.03310	0.03329
12	0.02788	0.02852	0.02907	0.02952	0.02989	0.03018
13	0.02492	0.02558	0.02616	0.02665	0.02707	0.02742
14	0.02239	0.02305	0.02363	0.02414	0.02459	0.02498
15	0.02022	0.02089	0.02144	0.02195	0.02241	0.02282
16	0.01833	0.01895	0.01952	0.02003	0.02049	0.02091
17	0.01669	0.01729	0.01783	0.01834	0.01879	0.01921
18	0.01526	0.01583	0.01636	0.01685	0.01729	0.01771
19	0.01399	0.01455	0.01505	0.01552	0.01596	0.01637
20	0.01289	0.01341	0.01389	0.01435	0.01477	0.01516
21	0.01189	0.01239	0.01286	0.01329	0.01370	0.01408
22	0.01102	0.01149	0.01193	0.01235	0.01275	0.01311
23	0.01023	0.01068	0.01110	0.01150	0.01188	0.01224
24	0.00953	0.00995	0.01036	0.01074	0.01110	0.01145
25	0.00889	0.00929	0.00968	0.01005	0.01039	0.01073

Remark: The statistical tables presented in Tables 4.1, 4.2 and 4.3 represent the table for the pdf of the Inverse Exponential distribution when the value of $x = 1, 2, 3, ..., 25$ and when the scale parameter ranges from 0.5 to 10. The values in the statistical tables were gotten using R software after an algorithm was developed for the density of the Inverse Exponential distribution. Details about the code have been provided in Appendix A. The table could be extended to include other values but for brevity, the range used were considered. From the tables, it was observed that as the value of x increases, the values for the pdf of the Inverse Exponential distribution increases and later decreases. This implies that the shape for the pdf of Inverse Exponential distribution is umimodal unlike the Exponential distribution that has a decreasing shape, this result also agrees with the work of Keller and Kamath (1982).

4.3 The Kumaraswamy Inverse Exponential Distribution

The detailed results for the Kumaraswamy Inverse Exponential (KIE) distribution are provided in this section.

Proposition 1 : Let X denote a non-negative continuous random variable such that; $X \sim KIE(a, b, \theta)$, then the cdf and pdf of the Kumaraswamy Inverse Exponential distribution are given as:

$$F(x) = 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b \tag{4.1}$$

and

$$f(x) = ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a\left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1} \tag{4.2}$$

respectively.

for $x > 0, a > 0, b > 0, \theta > 0$

where θ is a scale parameter, a and b are shape parameters.

Proof :

The cdf of the Kumaraswamy Generalised family of distributions is given as:

$$F(x) = 1 - \{1 - [G(x)]^a\}^b \tag{4.3}$$

The corresponding pdf is given as:

$$f(x) = abg(x)[G(x)]^{a-1}\{1 - [G(x)]^a\}^{b-1} \tag{4.4}$$

Recall from Equations (3.1) and (3.2) respectively that

$$G(x) = exp\left(-\frac{\theta}{x}\right)$$

and

$$g(x) = \frac{dG(x)}{dx}$$

Inserting $G(x)$ above which is also equivalent to Equation (3.1) into Equation (4.3) gives the cdf of the KIE distribution as:

$$F(x) = 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b$$

for $x > 0, a > 0, b > 0, \theta > 0$

Also, inserting Equations (3.1) and (3.2) into Equation (4.4) gives the pdf of the KIE distribution as:

$$f(x) = ab\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\left[exp\left(-\frac{\theta}{x}\right)\right]^{a-1}\left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1}$$

for $x > 0, a > 0, b > 0, \theta > 0$

This can further be simplified to give:

$$f(x) = ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a\left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1}$$

for $x > 0, a > 0, b > 0, \theta > 0$

Corollary 1 : The Kumaraswamy Inverse Exponential distribution is a valid distribution. This suffices that:

$$\int_0^\infty f(x)dx = 1$$

Or

$$\lim_{x\to\infty} F(x) = 1$$

For simplicity sake, $\lim_{x\to\delta} F(x) = 1$ is used in this study.

where $\delta \to \infty$.

Proof :

$$\lim_{x\to\delta} F(x) = \lim_{x\to\delta} \left\langle 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b\right\rangle$$
$$= \left\langle 1 - \left\{1 - \left[exp\left(-\frac{\theta}{\delta}\right)\right]^a\right\}^b\right\rangle$$

It is obvious that: $exp\left(-\frac{\theta}{\delta}\right) = 1$ since the division $\left(-\frac{\theta}{\delta}\right) \simeq 0$.

Therefore:

$$\lim_{x\to\delta} F(x) = \left\{1 - [1 - (1)^a]^b\right\}$$
$$= \left[1 - (1 - 1)^b\right]$$
$$= 1 - 0$$

Hence:

$$\lim_{x\to\infty} F(x) = 1$$

Therefore, $F(x) = 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b$ is a cdf.

4.3.1 Special Cases

Some distributions are found to be sub-models of the KIE distribution. For instance:

i When $b = 1$, the KIE distribution reduces to the GIE distribution that was proposed by Abouammoh and Alshingiti (2009).

ii When $a = b = 1$, the KIE distribution reduces to the Inverse Exponential distribution that was proposed by Keller and Kamath (1982).

The shape of the KIE distribution was investigated and established when its parameters were varied by the means of plots. The plots are displayed in Figure 4.1.

The shape of the KIE distribution was investigated at the following conditions: when all the three parameters are greater than one, when the values of the two shape parameters are greater than one while the scale parameter is less than one, when the value of one of the shape parameters (that is; a) is less than one while parameters b and θ are greater than one, and when the value of each of the parameters is less than one. However, the selected parameter values are for brevity purposes as plots at other parameter values produces similar shape(s).

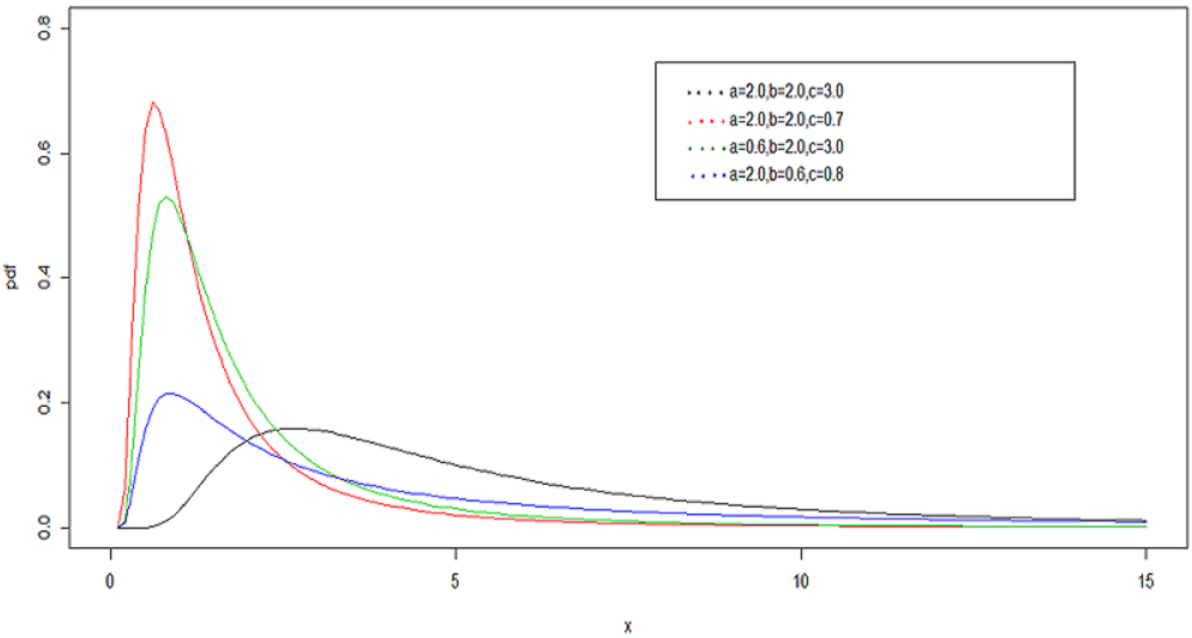

Figure 4.1: PDF of the KIE Distribution (where $c = \theta$)

The plot in Figure 4.1 gives an indication that the shape of the KIE distribution is unimodal. It can also be said that the distribution is positively skewed. Meanwhile, when the value of each of the parameters is less than one, the shape of the distribution decreases.

The graphical representation of the cdf of the KIE distribution is as shown in Figure 4.2. The plot at other parameter values produces a similar shape.

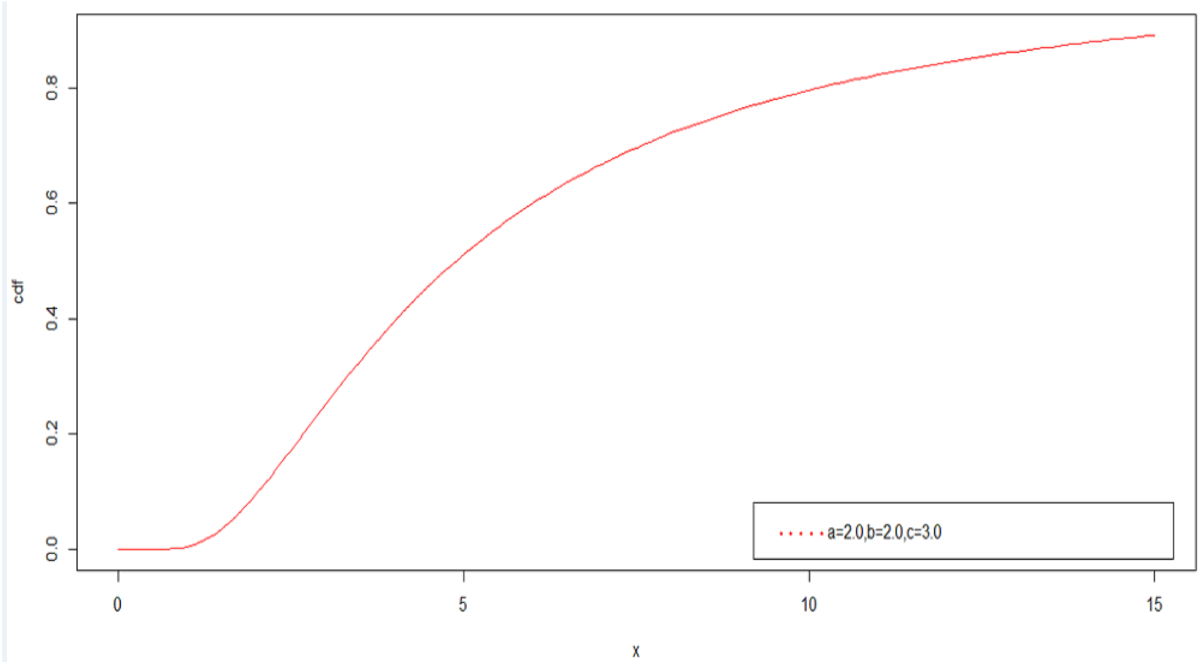

Figure 4.2: CDF of KIE Distribution $a = 2, b = 2, c = \theta = 3$

4.3.2 Reliability Analysis for Kumaraswamy Inverse Exponential Distribution

In this section, the expressions for the survival function and the hazard function of the KIE distribution are derived. As given in Equation (3.11), survival function is represented in mathematical form as:

$$S(x) = 1 - F(x)$$

In this context, $F(x)$ is the cdf of the KIE distribution as defined in Equation (4.1). Therefore, the expression for the survival function of the KIE distribution is given as:

$$S(x) = 1 - \left\langle 1 - \left\{ 1 - \left[exp\left(-\frac{\theta}{x} \right) \right]^{a} \right\}^{b} \right\rangle \tag{4.5}$$

Simplifying Equation (4.5) gives:

$$S(x) = \left\{ 1 - \left[exp\left(-\frac{\theta}{x} \right) \right]^{a} \right\}^{b} \quad ; \qquad x > 0, a > 0, b > 0, \theta > 0 \tag{4.6}$$

For brevity purpose, a graphical representation of the survival function of the KIE distribution at selected parameter values is shown in Figure 4.3:

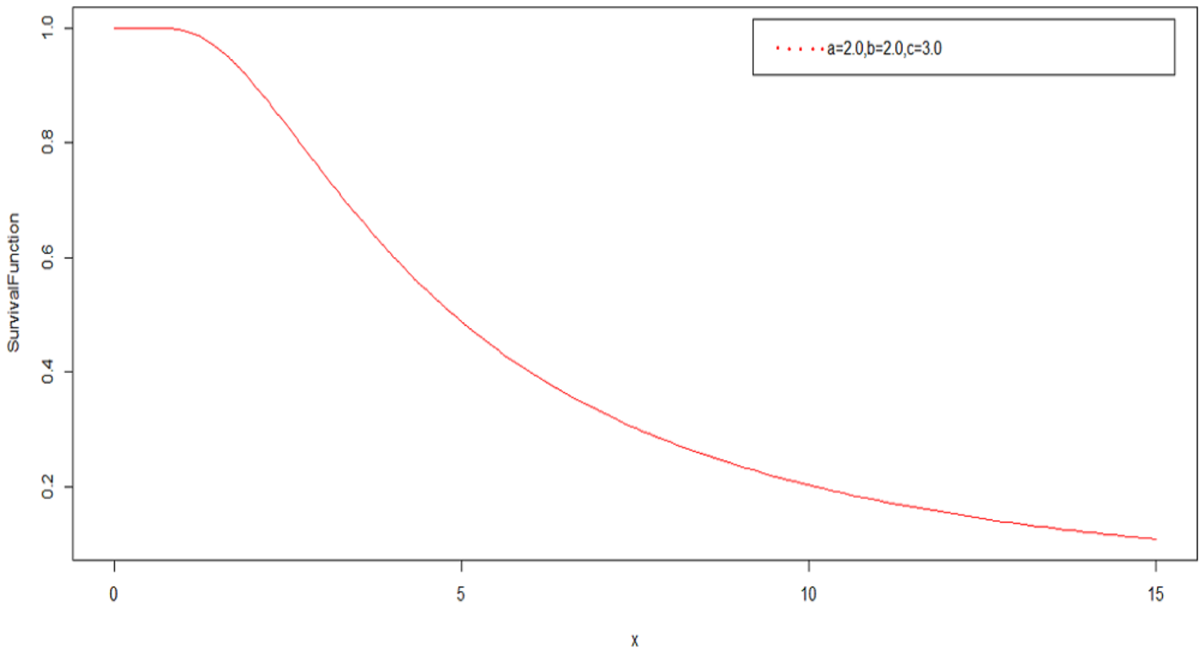

Figure 4.3: Survival Function of KIE Distribution at $a = 2, b = 2, c = \theta = 3$

Also, the hazard function was derived using:

$$h(x) = \frac{f(x)}{1 - F(x)}$$

In this context, $f(x)$ and $F(x)$ are the pdf and cdf of the KIE distribution respectively.

Then:

$$h(x) = \frac{ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1}}{\left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right\}^a\right\}^b} \tag{4.7}$$

This can be simplified as follows:

$$h(x) = \frac{ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b} \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{-1}}{\left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b}$$

$$h(x) = ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{-1} \tag{4.8}$$

Thus

$$h(x) = \frac{ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a}{\left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}} \quad ; \qquad x > 0, a > 0, b > 0, \theta > 0 \tag{4.9}$$

The shape of the hazard function of the KIE distribution is determined graphically at selected parameter values. These values are chosen for brevity purposes as plots at other parameter values would produce similar shape(s). The plots are as shown in Figure 4.4:

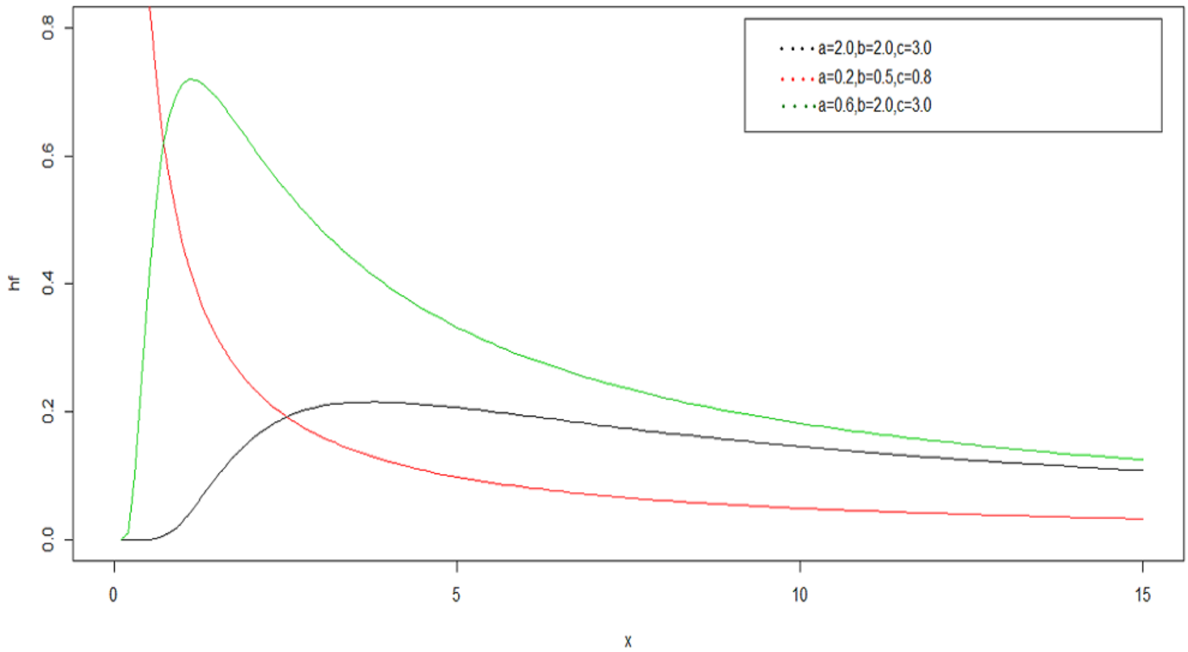

Figure 4.4: Hazard Function of KIE Distribution where; $c = \theta$

As shown in Figure 4.4, when the parameters are varied, it was observed that the shape of the hazard function of the KIE distribution shows either inverted bathtub or decreasing (reversed-J) shape.

4.3.3 Quantile Function and Median for Kumaraswamy Inverse Exponential Distribution

The quantile function is defined as the inverse of the cdf and it is given as: $Q(u) = F^{-1}(u)$.

Where

$$F(x) = 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^{a}\right\}^{b}$$

Set $F(x) = u$, then:

$$u = 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^{a}\right\}^{b}$$

$$1 - u = \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^{a}\right\}^{b}$$

$$(1-u)^{\frac{1}{b}} = 1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^{a}$$

$$1 - (1-u)^{\frac{1}{b}} = \left[exp\left(-\frac{\theta}{x}\right)\right]^{a}$$

$$\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}} = exp\left(-\frac{\theta}{x}\right)$$

$$\log\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}} = -\frac{\theta}{x}$$

$$x\left\{-\log\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\} = \theta$$

$$x = \theta\left\{-\log\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}^{-1} \tag{4.10}$$

Therefore, the quantile function of the KIE distribution is given as:

$$Q(u) = \theta\left\{-\log\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}^{-1} \tag{4.11}$$

Random numbers from the KIE distribution can be simulated using the expression in Equation (4.10) where $U \sim Uniform(0,1)$. In particular, the median of the KIE distribution can be derived by substituting $u = 0.5$ in Equation (4.11) as follows:

$$Median = \theta \left\{ -\log \left[1 - (1-0.5)^{\frac{1}{b}} \right]^{\frac{1}{a}} \right\}^{-1}$$

Therefore,

$$Median = \theta \left[-\log \left(1 - 0.5^{\frac{1}{b}} \right)^{\frac{1}{a}} \right]^{-1} \tag{4.12}$$

4.3.4 Order Statistics for the Kumaraswamy Inverse Exponential Distribution

Let $x_1, x_2, ..., x_n$ denote a random sample from a pdf $f(x)$ and an associated cdf $F(x)$ distributed according to the KIE distribution, then the pdf of kth order statistics of the KIE distribution was derived as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) \left[F(x)\right]^{k-1} \left[1 - F(x)\right]^{n-k}$$

Then,

$$f_{k:n}(x) = n \times \left\langle ab\frac{\theta}{x^2} \left[exp\left(-\frac{\theta}{x}\right)\right]^a \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1} \right\rangle \times \left\langle 1 - \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b \right\rangle^{k-1} \times \left\langle \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b \right\rangle^{n-k} \tag{4.13}$$

Therefore, the distribution of minimum order statistics is given as:

$$f_{1:n}(x) = n \times \left\langle ab\frac{\theta}{x^2} \left[exp\left(-\frac{\theta}{x}\right)\right]^a \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1} \right\rangle \times \left\langle \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b \right\rangle^{n-1} \tag{4.14}$$

This was gotten by substituting $k = 1$ in Equation (4.13).

In the same way, the distribution of maximum order statistics for the KIE distribution is given as:

$$f_{n:n}(x) = n \times \left\langle ab\frac{\theta}{x^2}\left[exp\left(-\frac{\theta}{x}\right)\right]^a\left\{1-\left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^{b-1}\right\rangle \times$$

$$\left\langle 1-\left\{1-\left[exp\left(-\frac{\theta}{x}\right)\right]^a\right\}^b\right\rangle^{n-1} \qquad (4.15)$$

This was gotten by substituting $k = n$ in Equation (4.13).

4.3.5 Parameter Estimation

To estimate the parameters of the KIE distribution, the method of Maximum Likelihood Estimation (MLE) was used as follows; Let $X = x_1, x_2, ..., x_n$ be a random sample of n independently and identically distributed random variables each having the KIE distribution, the likelihood function is given as:

$$L(x_1, x_2, ..., x_n; a, b, \theta) = f(x_1, x_2, ..., x_n; a, b, \theta) = \prod_{i=1}^{n} f\left(x_i; a, b, \theta\right)$$

$$L(x_1, x_2, ..., x_n; a, b, \theta) = \prod_{i=1}^{n}\left\langle ab\frac{\theta}{x_i^2}\left[exp\left(-\frac{\theta}{x_i}\right)\right]^a\left\{1-\left[exp\left(-\frac{\theta}{x_i}\right)\right]^a\right\}^{b-1}\right\rangle$$

$$L(x_1, x_2, ..., x_n; a, b, \theta) = (ab\theta)^n \times \sum_{i=1}^{n} x_i^{-2} \times exp\left[-a\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right] \times$$

$$\sum_{i=1}^{n}\left[1-exp\left(-a\theta x_i^{-1}\right)\right]^{b-1} \qquad (4.16)$$

Take the natural logarithm of $L(x_1, x_2, ..., x_n; a, b, \theta)$

Also, let $l = \log L(x_1, x_2, ..., x_n; a, b, \theta)$, then

$$l = n\log(a) + n\log(b) + n\log(\theta) - 2\sum_{i=1}^{n}\log(x_i) - a\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) +$$

$$(b-1)\sum_{i=1}^{n}\log[1-exp(-a\theta x_i^{-1})]$$

Differentiating l with respect to parameters a, b and θ respectively gives:

$$\frac{dl}{da} = \frac{n}{a} - \theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + (b-1)\sum_{i=1}^{n}\frac{\theta x_i^{-1}exp(-a\theta x_i^{-1})}{[1-exp(-a\theta x_i^{-1})]} \qquad (4.17)$$

$$\frac{dl}{db} = \frac{n}{b} + \sum_{i=1}^{n}\log[1-exp(-a\theta x_i^{-1})] \qquad (4.18)$$

$$\frac{dl}{d\theta} = \frac{n}{\theta} - a\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + (b-1)\sum_{i=1}^{n}\frac{ax_i^{-1}exp(-a\theta x_i^{-1})}{[1-exp(-a\theta x_i^{-1})]} \tag{4.19}$$

Solving $\frac{dl}{da} = 0, \frac{dl}{db} = 0$ and $\frac{dl}{d\theta} = 0$ simultaneously gives the maximum likelihood estimates of parameters a, b and θ respectively. Meanwhile, the result cannot be gotten analytically except numerically from suitable statistical software when data sets are available. In this study, the result is obtained directly from the available data sets using R software.

Next, the application of the KIE distribution using the available data sets in Chapter three of this study is considered.

4.3.6 Applications of Kumaraswamy Inverse Exponential Distribution to Data sets

The KIE distribution was applied to six real life data sets in order to assess its statistical superiority over its sub-models; the Generalised Inverse Exponential (GIE) distribution and Inverse Exponential (IE) distribution. The data sets used for this analysis are DATA I to DATA VI. The data sets have been described in Chapter three of this study and the summaries are given as follows:

Starting with DATA I, the summary of the data is given in Table 4.4:

Table 4.4: Summary of data on failure and running times of devices

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
30	0.0200	3.0000	1.9650	1.7700	1.3223	-0.2699	1.4537

It can be noticed from Table 4.4 that the data set is slightly negatively skewed with the coefficient of skewness being -0.2699 and a variance of 1.3223.

The performance of the KIE distribution with respect to the GIE distribution and the IE distribution using the observations in DATA I is as shown in Table 4.5:

Table 4.5: Performance Ratings of KIE distribution Using DATA I

Distribution	Parameters	Log-Likelihood	AIC	Rank
KIE	$\widehat{a} = 44.5386$ $\widehat{b} = 0.4953$ $\widehat{\theta} = 0.0040$	-65.0736	136.1471	1
GIE	$\widehat{a} = 34.2239$ $\widehat{\theta} = 0.0091$	-70.6309	145.2617	3
IE	$\widehat{\theta} = 0.3122$	-70.6309	143.2617	2

Remark : The distribution that corresponds to the lowest AIC or highest log-likelihood is considered the best fit.

From Table 4.5, the KIE distribution has the highest log-likelihood value of -65.0736 and the lowest AIC value of 136.1471, therefore, the KIE distribution provides a better fit than the GIE and IE distributions.

For DATA II, the summary is given in Table 4.6:

Table 4.6: Summary of data on relief times of patients receiving an analgesic

n	**Min.**	**Max.**	**Median**	**Mean**	**Var.**	**Skewness**	**Kurtosis**
20	1.100	4.100	1.700	1.900	0.4958	1.5924	5.9241

It can be noticed from Table 4.6 that the data set is positively skewed with the coefficient of skewness being 1.5924 and a variance of 0.4958.

The performance of the KIE distribution with respect to the GIE distribution and the IE distribution using the observations in DATA II is shown in Table 4.7:

Table 4.7: Performance Ratings of KIE distribution Using DATA II

Distribution	**Parameters**	**Log-Likelihood**	**AIC**	**Rank**
KIE	$\widehat{a} = 96.8835$ $\widehat{b} = 20.7662$ $\widehat{\theta} = 0.0637$	-17.1046	40.2091	1
GIE	$\widehat{a} = 0.0031$ $\widehat{\theta} = 66.7625$	-57.0997	118.1993	3
IE	$\widehat{\theta} = 1.7247$	-32.6687	67.3373	2

Remark : From Table 4.7, the KIE distribution has the highest log-likelihood value of -17.1046 and the lowest AIC value of 40.2091, therefore, the KIE distribution fits the data set better than the GIE and IE distributions.

For DATA III, the summary is given in Table 4.8:
It can be noticed from Table 4.8 that the data set is positively skewed with the coefficient of skewness being 2.1702 and a variance of 4,925.205.

Table 4.8: Summary of data on death times of patients with cancer of the tongue

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
52	1.00	400.00	77.00	80.73	4,925.205	2.1702	10.0714

The performance of the KIE distribution with respect to the GIE distribution and the IE distribution using the observations in DATA III is shown in Table 4.9:

Table 4.9: Performance Ratings of KIE distribution Using DATA III

Distribution	Parameters	Log-Likelihood	AIC	Rank
KIE	$\widehat{a} = 0.5138$ $\widehat{b} = 0.5771$ $\widehat{\theta} = 22.0303$	-300.856	607.712	1
GIE	$\widehat{a} = 0.0244$ $\widehat{\theta} = 713.67$	-306.1066	616.2133	3
IE	$\widehat{\theta} = 17.3795$	-306.1066	614.2133	2

Remark : From Table 4.9, the KIE distribution has the highest log-likelihood value of -300.856 and the lowest AIC value of 607.712, therefore, the KIE distribution fits the data set better than the GIE and IE distributions.

For DATA IV, the summary is given in Table 4.10:

Table 4.10: Summary of data on failure times of the air conditioning system of an airplane

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
30	1.0	261.0	22.0	59.6	5,167.421	1.609	4.967

It can be noticed from Table 4.10 that the data set is positively skewed with the coefficient of skewness being 1.609 and a variance of 5,167.421.

The performance of the KIE distribution with respect to the GIE distribution and the IE distribution using the observations in DATA IV is shown in Table 4.11:

Table 4.11: Performance Ratings of KIE distribution Using DATA IV

Distribution	Parameters	Log-Likelihood	AIC	Rank
KIE	$\widehat{a} = 0.2162$ $\widehat{b} = 0.6493$ $\widehat{\theta} = 37.5806$	-159.1929	320.3859	3
GIE	$\widehat{a} = 0.0093$ $\widehat{\theta} = 33.6974$	-70.6309	145.2617	1
IE	$\widehat{\theta} = 11.1800$	-159.0620	320.1239	2

Remark : From Table 4.11, the GIE distribution has the highest log-likelihood value of -70.6309 and the lowest AIC value of 145.2617, therefore, the GIE distribution fits the data set better than the KIE and IE distributions. Infact, out of the three competing distributions, the performance of the KIE distribution is the poorest.

For DATA V, the summary is given in Table 4.12:

Table 4.12: Summary of data on death times of patients with cancer of the tongue

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
72	12.00	376.00	70.00	99.82	6,580.122	1.7589	5.6144

It can be noticed from Table 4.12 that the data set is positively skewed with the coefficient of skewness being 1.7589 and a variance of 6,580.122.

The performance of the KIE distribution with respect to the GIE distribution and the IE distribution using the observations in DATA V is shown in Table 4.13:

Table 4.13: Performance Ratings of KIE distribution Using DATA V

Distribution	Parameters	Log-Likelihood	AIC	Rank
KIE	$\widehat{a} = 8.0442$ $\widehat{b} = 2.5424$ $\widehat{\theta} = 12.7589$	-391.5948	788.1897	2
GIE	$\widehat{a} = 0.0249$ $\widehat{\theta} = 29.9943$	-98.4873	200.9745	1
IE	$\widehat{\theta} = 60.0980$	-402.6718	807.3437	3

Remark : From Table 4.13, the GIE distribution has the highest log-likelihood value of -98.4873 and the lowest AIC value of 200.9745, therefore, the GIE distribution fits the data set better than the KIE and IE distributions.

For DATA VI, the summary is given in Table 4.14:

Table 4.14: Summary of data on survival times of patients suffering from head and neck cancer

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
44	12.20	1,776.00	128.50	223.50	93,286.41	3.2691	16.5596

It can be noticed from Table 4.14 that the data set is positively skewed with the coefficient of skewness being 3.2691 and a variance of 93,286.41.

The performance of the KIE distribution with respect to the GIE distribution and the IE distribution using the observations in DATA VI is shown in Table 4.15:

Table 4.15: Performance Ratings of KIE distribution Using DATA VI

Distribution	Parameters	Log-Likelihood	AIC	Rank
KIE	$\widehat{a} = 57.1140$ $\widehat{b} = 1.1679$ $\widehat{\theta} = 1.4856$	-279.8058	564.6116	3
GIE	$\widehat{a} = 0.1613$ $\widehat{\theta} = 28.3982$	-84.0759	172.1519	1
IE	$\widehat{\theta} = 76.7000$	-279.5773	561.1546	2

Remark : From Table 4.15, the GIE distribution has the highest log-likelihood value of -84.0759 and the lowest AIC value of 172.1519, therefore, the GIE distribution fits the data set better than the KIE and IE distributions.

4.4 The Transmuted Inverse Exponential (TIE) Distribution

In this section, detailed results for the Transmuted Inverse Exponential (TIE) distribution are provided.

Proposition 2 : Let X denote a non-negative continuous random variable such that; $X \sim TIE(\theta, \lambda)$, then the cdf and pdf of the Transmuted Inverse Exponential distribution are given as:

$$F(x) = \left[exp\left(-\frac{\theta}{x}\right)\right]\left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right] \tag{4.20}$$

and

$$f(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)\left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right] \tag{4.21}$$

respectively.
for $x > 0, \theta > 0$ and $|\lambda| \leq 1$

where θ is a scale parameter and λ is the transmuted parameter.

Proof :

The cdf of the TIE distribution was obtained from the relation:

$$F(x) = (1 + \lambda)\, G(x) - \lambda[G(x)]^2 \tag{4.22}$$

for $|\lambda| \leq 1$,

Recall from Equations (3.1) and (3.2) respectively that

$$G(x) = exp\left(-\frac{\theta}{x}\right)$$

and

$$g(x) = \frac{dG(x)}{dx}$$

Inserting $G(x)$ above which is also equivalent to Equation (3.1) into Equation (4.22) gives the cdf of the TIE distribution as:

$$F(x) = \left[exp\left(-\frac{\theta}{x}\right)\right]\left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]$$

for $x > 0, \theta > 0$ and $|\lambda| \leq 1$

Its pdf is obtained from the relation:

$$f(x) = g(x)\left[1 + \lambda - 2\lambda G(x)\right] \tag{4.23}$$

for $|\lambda| \leq 1$

where $G(x)$ and $g(x)$ are the cdf and pdf of the Inverse Exponential distribution as defined in Equations (3.1) and (3.2) respectively.

Hence, the pdf of the TIE distribution is given as:

$$f(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)\left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right] \tag{4.24}$$

for $x > 0, \theta > 0$ and $|\lambda| \leq 1$

Corollary 2 : The Transmuted Inverse Exponential distribution is a valid distribution. This suffices that:

$$\lim_{x \to \delta} F(x) = 1$$

where $\delta \to \infty$.

Proof :

$$\lim_{x \to \delta} F(x) = \lim_{x \to \delta}\left\{\left[exp\left(-\frac{\theta}{x}\right)\right]\left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]\right\}$$

$$= \left[exp\left(-\frac{\theta}{\delta}\right)\right]\left[1 + \lambda - \lambda exp\left(-\frac{\theta}{\delta}\right)\right]$$

where $exp\left(-\frac{\theta}{\delta}\right) \simeq 1$, since $\delta \to \infty$.

Therefore,

$$\lim_{x \to \delta} F(x) = 1 \times [1 + \lambda - \lambda(1)]$$

Hence:

$$\lim_{x \to \infty} F(x) = 1$$

Thus, the expression in Equation (4.20) is a cdf and by extension, Equation (4.21) is a pdf.

4.4.1 Special Cases

Some distributions are found to be sub-models of the proposed TIE distribution. For instance:

i When $\lambda = 0$, the TIE distribution reduces to give the Inverse Exponential distribution that was proposed by Keller and Kamath (1982).
That is,

$$f(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)\left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right]$$
$$f(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)\left[1 + 0 - 2(0) exp\left(-\frac{\theta}{x}\right)\right]$$
$$f(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)$$

which is the pdf of Inverse Exponential distribution.

ii If a random variable Y is such that; $Y = \frac{1}{X}$ and $\lambda = 0$, then the TIE distribution reduces to give the Exponential distribution.

The possible shape(s) of the pdf of TIE distribution are determined and established graphically when its parameter values were varied. These values were selected for brevity purposes as plots at other parameter values would produce similar shape(s). The plots are as shown in Figure 4.5:

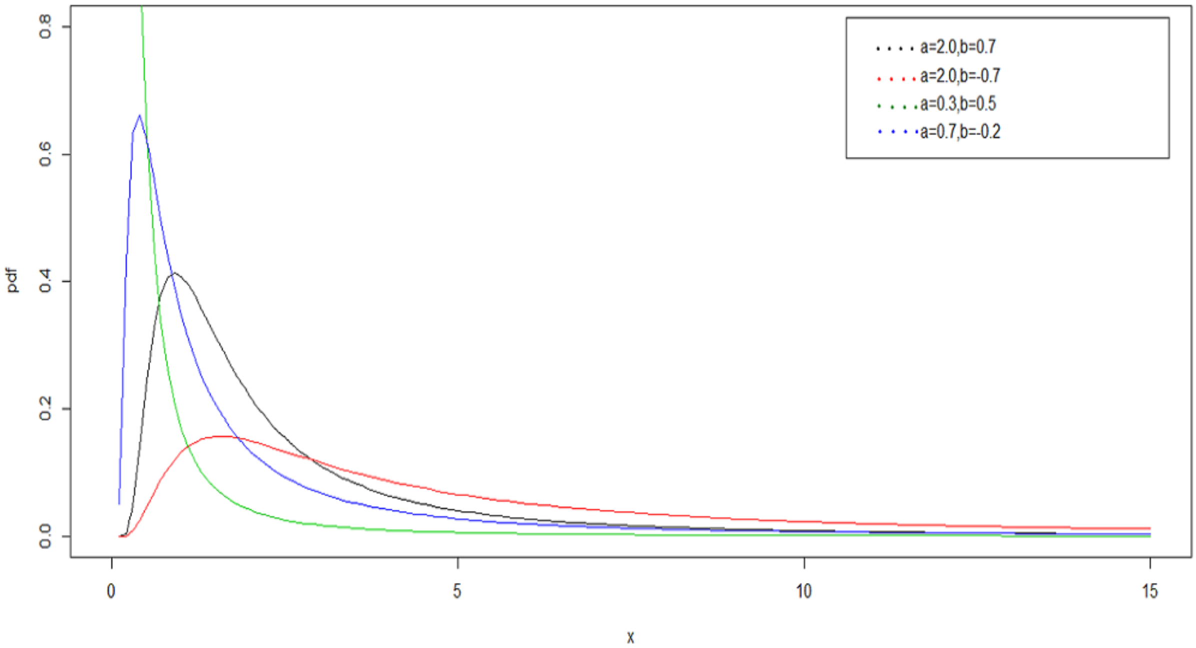

Figure 4.5: PDF of TIE Distribution where $a = \theta, b = \lambda$

Obviously, the shape of the TIE distribution is unimodal as shown in Figure 4.5 except when the value of the scale parameter and the transmuted parameter are less than one, it was discovered that the shape could also be decreasing (or reversed J-shape).

The graphical representation of the cdf of the TIE distribution at selected and arbitrary parameter values of θ and λ is as shown in Figure 4.6:

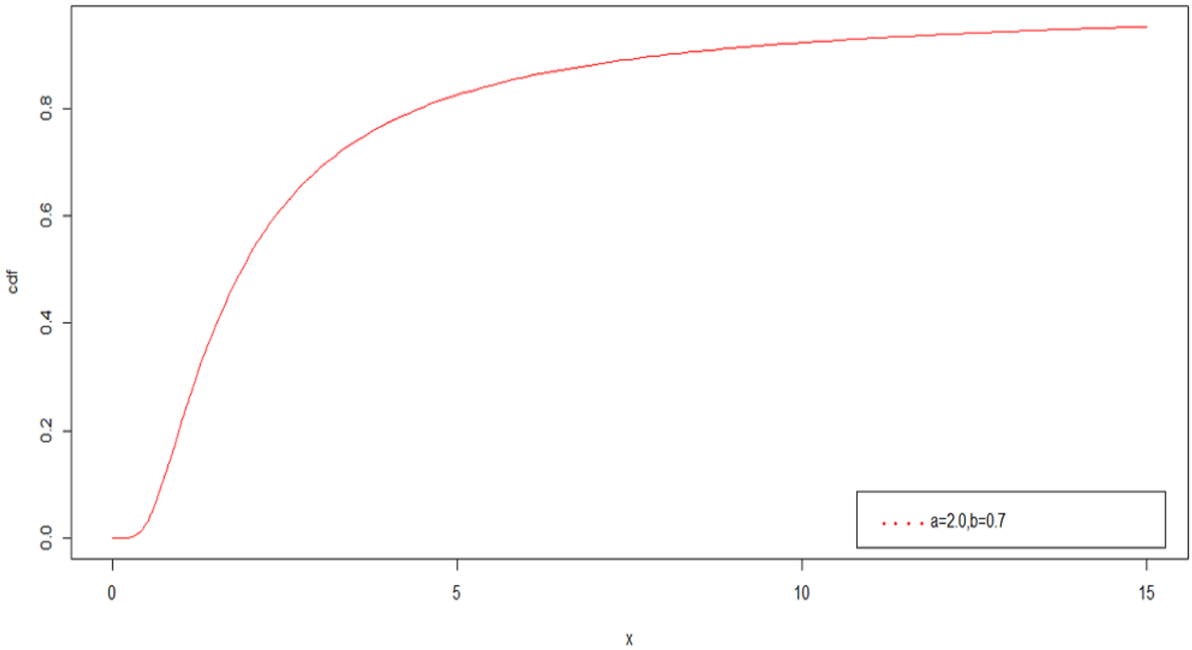

Figure 4.6: CDF of TIE Distribution at $a = \theta = 2, b = \lambda = 0.7$

4.4.2 Reliability Analysis for the TIE Distribution

The general formula for the survival function is represented as:

$$S(x) = Pr(X > x) = 1 - F(x)$$

Where, $F(x)$ is the cdf of the TIE distribution. Therefore, the survival function of the TIE distribution is obtained as:

$$S(x) = 1 - \left[exp\left(-\frac{\theta}{x}\right)\right]\left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right] \tag{4.25}$$

for $x > 0, \theta > 0$ and $|\lambda| \leq 1$

For brevity, a graphical display of the survival function of the TIE distribution at selected parameter values is shown in Figure 4.7:

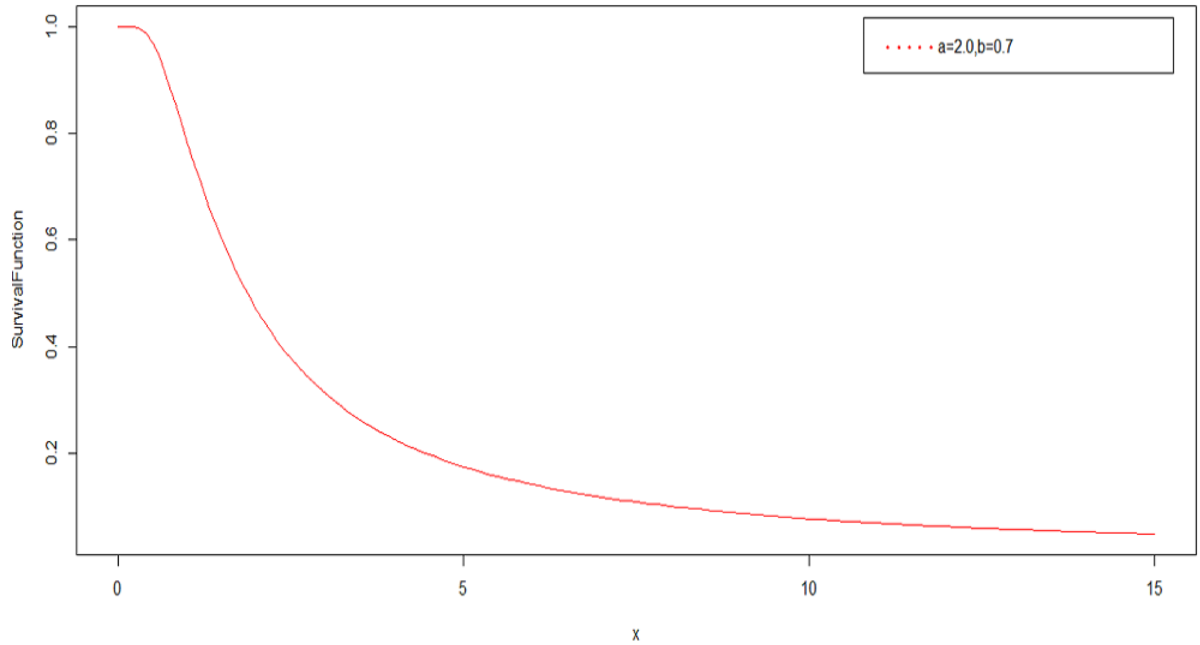

Figure 4.7: Survival Function of TIE Distribution at $a = \theta = 2, b = \lambda = 0.7$

The Hazard function on the other hand is mathematically given as:

$$h(x) = \frac{f(x)}{1 - F(x)}$$

In this context, $f(x)$ and $F(x)$ are the pdf and cdf of the TIE distribution respectively. Therefore, the hazard function for the TIE distribution can be expressed as:

$$h(x) = \frac{\frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)\left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right]}{1 - \left[exp\left(-\frac{\theta}{x}\right)\right]\left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]} \tag{4.26}$$

for $x > 0, \theta > 0$ and $|\lambda| \leq 1$

The graphical representation of the hazard function of the TIE distribution at various selected parameter values of θ and λ are given in Figure 4.8. These parameter values are selected for brevity purposes:

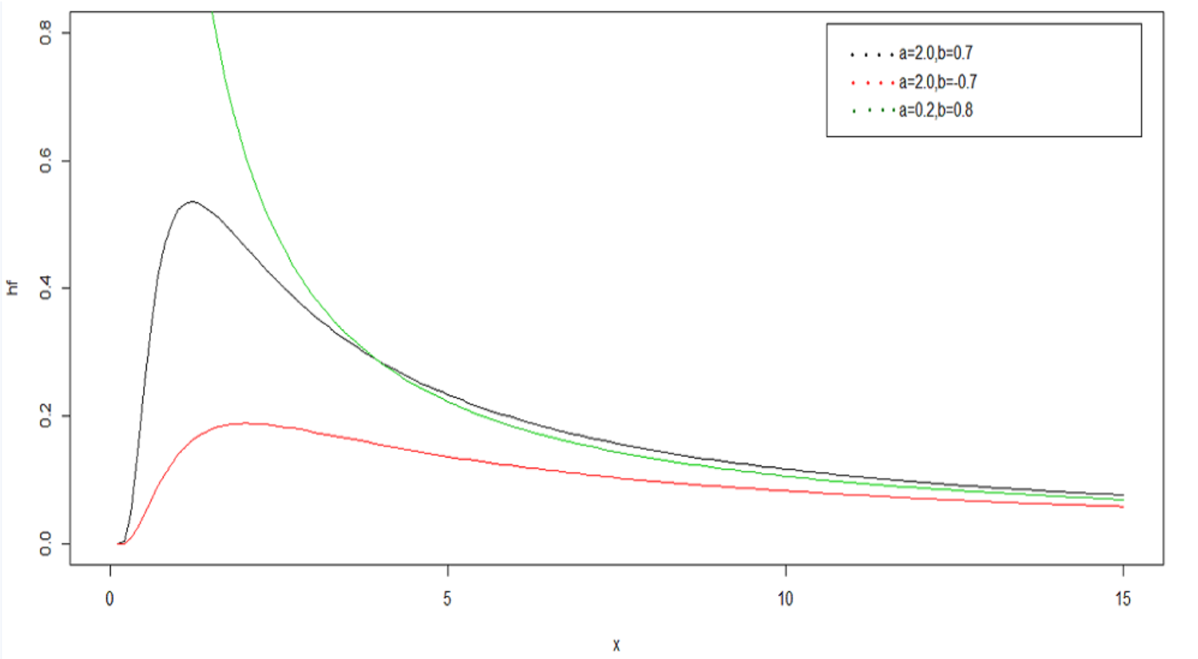

Figure 4.8: Hazard Function of TIE Distribution where; $a = \theta, b = \lambda$

Plots at several selected values of the parameters show that the shape of the failure rate of the TIE distribution is either inverted bathtub or reversed J-shape.

4.4.3 Quantile Function and Median for Transmuted Inverse Exponential Distribution

The general formula for the quantile function is:

$$Q(u) = F^{-1}(u)$$

Where $F^{-1}(u)$ is the inverse of the cdf of the TIE distribution.

The quantile function; $Q(u)$ of the TIE distribution was obtained directly from the work of Khan *et al.*, (2014) who gave the quantile function of the Transmuted Inverse Weibull distribution as:

$$Q(u) = \theta \left[\log \left(\frac{2\lambda}{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}} \right) \right]^{-\frac{1}{\alpha}} \tag{4.27}$$

When $\alpha = 1$ in Equation (4.27), then the resulting equation gives the quantile function of the TIE distribution as:

$$Q(u) = \theta \left[\log \left(\frac{2\lambda}{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}} \right) \right]^{-1} \tag{4.28}$$

Therefore, the median of the TIE distribution can be obtained by substituting $u = 0.5$ into Equation (4.28) and it is given as:

$$Median = \theta \left[\log \left(\frac{2\lambda}{(1+\lambda) - \sqrt{(1+\lambda)^2 - 2\lambda}} \right) \right]^{-1}$$

This can be simplified to give:

$$Median = \theta \left[\log \left(\frac{2\lambda}{(1+\lambda) - \sqrt{1+\lambda^2}} \right) \right]^{-1} \tag{4.29}$$

For simulation purposes, samples from the TIE distribution can be obtained using the expression:

$$x = \theta \left[\log \left(\frac{2\lambda}{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}} \right) \right]^{-1} \tag{4.30}$$

where $U \sim Uniform(0, 1)$

4.4.4 Distribution of Order Statistics for Transmuted Inverse Exponential Distribution

Let $x_1, x_2, ..., x_n$ denote a random sample from a pdf $f(x)$ and an associated cdf $F(x)$ distributed according to the TIE distribution, then the pdf of kth order statistics of the TIE distribution was derived as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) \left[F(x)\right]^{k-1} \left[1 - F(x)\right]^{n-k}$$

$$\begin{aligned} f_{k:n}(x) = n \times & \left\{ \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right) \left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right]\right\} \times \\ & \left\{ \left[exp\left(-\frac{\theta}{x}\right)\right] \left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]\right\}^{k-1} \times \\ & \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right] \left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]\right\}^{n-k} \end{aligned} \quad (4.31)$$

Therefore, the distribution of minimum order statistics for the TIE distribution is given as:

$$\begin{aligned} f_{1:n}(x) = n \times & \left\{ \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right) \left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right]\right\} \times \\ & \left\{1 - \left[exp\left(-\frac{\theta}{x}\right)\right] \left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]\right\}^{n-1} \end{aligned} \quad (4.32)$$

In the same way, the distribution of maximum order statistics for the TIE distribution is given as:

$$\begin{aligned} f_{n:n}(x) = n \times & \left\{ \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right) \left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x}\right)\right]\right\} \times \\ & \left\{ \left[exp\left(-\frac{\theta}{x}\right)\right] \left[1 + \lambda - \lambda exp\left(-\frac{\theta}{x}\right)\right]\right\}^{n-1} \end{aligned} \quad (4.33)$$

4.4.5 Parameter Estimation for the Transmuted Inverse Exponential Distribution

To estimate the two parameters of the TIE distribution, the method of maximum likelihood estimation (MLE) was used as follows; Let $X = x_1, x_2, ..., x_n$ be a random sample of size n independently and identically distributed random variables each distributed according to the TIE distribution, the likelihood function is given as:

$$L(x_1, x_2, ..., x_n; \theta, \lambda) = f(x_1, x_2, ..., x_n; \theta, \lambda) = \prod_{i=1}^{n} f\left(x_i|\theta, \lambda\right)$$

$$L(x_1, x_2, ..., x_n; \theta, \lambda) = \prod_{i=1}^{n} \left\{ \frac{\theta}{x_i^2} exp\left(-\frac{\theta}{x_i}\right) \left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x_i}\right)\right]\right\}$$

Let $l = \log L(x_1, x_2, ..., x_n; \theta, \lambda)$,

$$l = n \log \theta - 2\sum_{i=1}^{n} \log(x_i) - \sum_{i=1}^{n} \left(\frac{\theta}{x_i}\right) + \sum_{i=1}^{n} \log \left[1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x_i}\right)\right]$$

Differentiating l with respect to each of the two parameters; θ and λ respectively gives:

$$\frac{dl}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \left(\frac{1}{x_i}\right) + 2\lambda \sum_{i=1}^{n} \frac{\left(\frac{1}{x_i}\right) exp\left(-\frac{\theta}{x_i}\right)}{1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x_i}\right)}$$

$$\frac{dl}{d\lambda} = \sum_{i=1}^{n} \frac{1 - 2exp\left(-\frac{\theta}{x_i}\right)}{1 + \lambda - 2\lambda exp\left(-\frac{\theta}{x_i}\right)}$$

Equating $\frac{dl}{d\theta}$ and $\frac{dl}{d\lambda}$ to zero and solving the resulting equations give the maximum likelihood estimates of parameters θ and λ respectively. The results cannot be obtained analytically but it can be gotten numerically. Hence, R software is used in this study to estimate the parameters with respect to the available data sets.

Next, the applications of the TIE distribution to the real life data sets given in Chapter three are presented and discussed.

4.4.6 Applications of TIE Distribution to Data sets

To demonstrate the usefulness and flexibility of the TIE distribution, DATA VII, VIII and VI are used.

For DATA VII, the summary is given in Table 4.16:

Table 4.16: Summary of data on waiting times (in mins.) of bank customers

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
100	0.800	38.500	4.675	13.020	52.37411	1.472765	5.540292

The TIE distribution was compared with the IE distribution using the information in DATA VII, the performances of the distributions under study are given in Table 4.17:

Table 4.17: Performance Ratings of TIE distribution Using DATA VII

Distribution	Parameters	Log-Likelihood	AIC	Rank
TIE	$\widehat{\theta} = 10.7924$ $\widehat{\lambda} = 1.8755$	-323.2699	650.5397	1
IE	$\widehat{\theta} = 5.3476$	-336.5585	675.117	2

Remark : The TIE distribution has the highest log-likelihood value of -323.2699 and the lowest AIC value of 650.5397. Hence, the TIE distribution is considered to be better than the IE distribution for the data set considered.

For DATA VIII, the summary is given in Table 4.18:

Table 4.18: Summary of data on vinyl chloride

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
34	0.100	8.000	0.500	2.475	3.812594	1.603688	5.005408

The TIE distribution was compared with the IE distribution using the information in DATA VIII, the performances of the distributions under study are given in Table 4.19:

Table 4.19: Performance Ratings of TIE distribution Using DATA VIII

Distribution	Parameters	Log-Likelihood	AIC	Rank
TIE	$\widehat{\theta} = 0.4138$ $\widehat{\lambda} = -0.6301$	-57.92018	119.8404	1
IE	$\widehat{\theta} = 0.57247$	-59.19302	120.386	2

Remark : The TIE distribution has the highest log-likelihood value of -57.92018 and the lowest AIC value of 119.8404, hence the TIE distribution is considered to be better than the IE distribution for the data set used.

Next, DATA VI which was considered to be over-dispersed to know if TIE distribution would still be better than its baseline distribution. Meanwhile, the summary of DATA VI has been given in Table 4.14.

The performances of the TIE and IE distributions with respect to the information in DATA VI are given in Table 4.20:

Table 4.20: Performance Ratings of TIE distribution Using DATA VI

Distribution	Parameters	Log-Likelihood	AIC	Rank
TIE	$\widehat{\theta} = 49.6008$ $\widehat{\lambda} = -0.7164$	-279.9608	562.3217	2
IE	$\widehat{\theta} = 76.703$	-279.5773	561.1546	1

Remark : Based on the log-likelihood and AIC values in Table 4.20, it can be seen that the IE distribution performs better than the TIE distribution.

4.5 The Exponentiated Generalised Inverse Exponential Distribution

Next, the densities for the Exponentiated Generalised Inverse Exponential distribution are provided.

Proposition 3 : Let X denote a non-negative continuous random variable such that; $X \sim EGIE(\alpha, \beta, \theta)$, then its cdf is given as:

$$F(x) = \left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta} \tag{4.34}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

The corresponding pdf is given as:

$$f(x) = \alpha\beta\theta x^{-2} exp\left(-\frac{\theta}{x}\right)\left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha-1}\left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta-1} \tag{4.35}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$.

where α and β are shape parameters and θ is a scale parameter.

Proof :

The cdf of the Exponentiated Generalised Inverse Exponential distribution was derived from:

$$F(x) = \{1 - [1 - G(x)]^{\alpha}\}^{\beta} \tag{4.36}$$

where $\alpha > 0$ and $\beta > 0$ are additional shape parameters.

In this context, $G(x)$ and $g(x)$ are the cdf and pdf of the Inverse Exponential distribution as defined in Equations (3.1) and (3.2) respectively.

That is,

$$G(x) = exp\left(-\frac{\theta}{x}\right)$$

and

$$g(x) = \frac{\theta}{x^2} exp\left(-\frac{\theta}{x}\right)$$

respectively.

With this understanding, inserting Equation (3.1) into Equation (4.36) gives the cdf of the Exponential Generalised Inverse Exponential (EGIE) distribution as:

$$F(x) = \left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

Its pdf was derived from:

$$f(x) = \alpha\beta g(x)\left[1 - G(x)\right]^{\alpha-1}\left\{1 - \left[1 - G(x)\right]^{\alpha}\right\}^{\beta-1} \tag{4.37}$$

Now, inserting Equations (3.1) and (3.2) into Equation (4.37) gives the pdf of the Exponentiated Generalised Inverse Exponential distribution as:

$$f(x) = \alpha\beta\theta x^{-2} exp\left(-\frac{\theta}{x}\right)\left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha-1}\left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta-1} \tag{4.38}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

Corollary 3 : The Exponentiated Generalised Inverse Exponential distribution is a valid distribution. It suffices that:

$$\lim_{x\to\delta} F(x) = 1$$

where $\delta \to \infty$.

In this context, $F(x)$ is the cdf of the EGIE distribution.

Proof :

$$\lim_{x\to\delta} F(x) = \lim_{x\to\delta}\left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}$$

$$= \left\{1 - \left[1 - exp\left(-\frac{\theta}{\delta}\right)\right]^{\alpha}\right\}^{\beta}$$

Meanwhile, $exp\left(-\frac{\theta}{\delta}\right) \simeq 1$, since $\delta \to \infty$. Therefore,

$$\lim_{x \to \delta} F(x) = [1 - (1 - 1)^{\alpha}]^{\beta}$$

$$= (1 - 0)^{\beta}$$

$$= 1$$

This completes the proof.

4.6 Special Cases

Some distributions have been discovered to be sub-models of the EGIE distribution. For instance:

i When $\beta = 1$, the EGIE distribution reduces to give the Generalised Inverse Exponential distribution introduced by Abouammoh and Alshingiti (2009).

ii When $\alpha = \beta = 1$, the EGIE distribution reduces to give the baseline distribution (Inverse Exponential distribution) which was introduced by Keller and Kamath (1982).

The graphical representation of the pdf of the EGIE distribution at various selected parameter values are provided in Figure 4.9. Meanwhile, plots at other parameter values would produce similar shape(s):

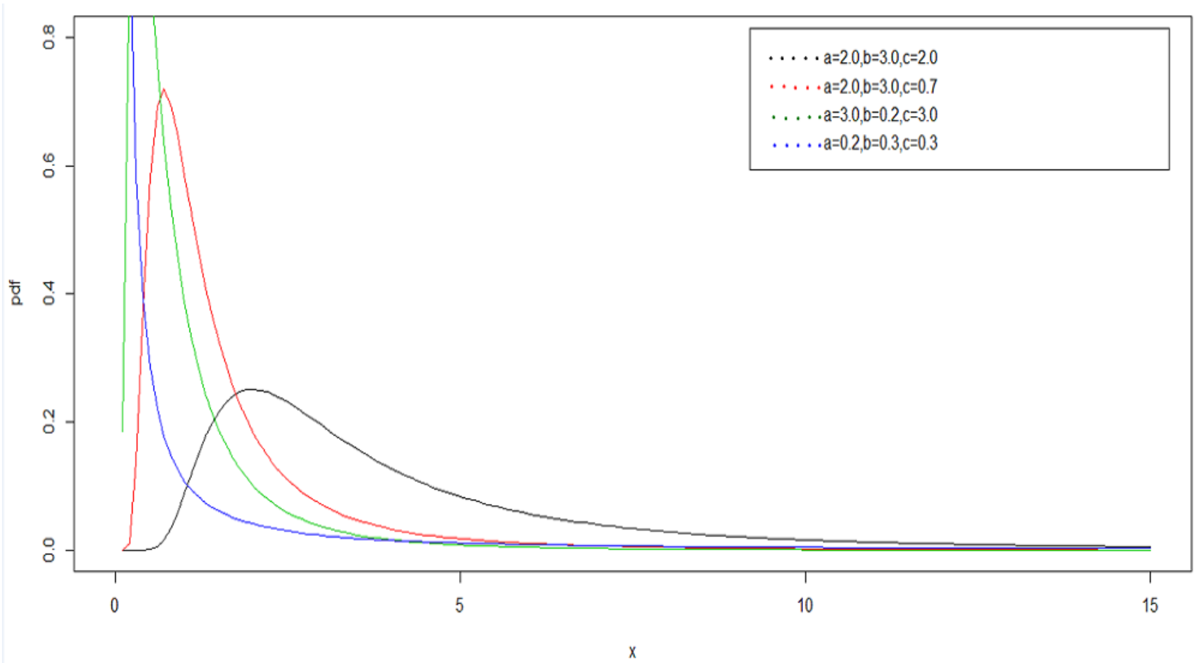

Figure 4.9: PDF of EGIE Distribution where; $a = \alpha, b = \beta, c = \theta$

The plots in Figure 4.9 show that the shape of the EGIE distribution exhibits unimodality (inverted bathtub shape) and could also be decreasing (depending on the parameter values).

For brevity purposes, a graphical display of the cdf of the EGIE distribution at selected parameter values is shown in Figure 4.10:

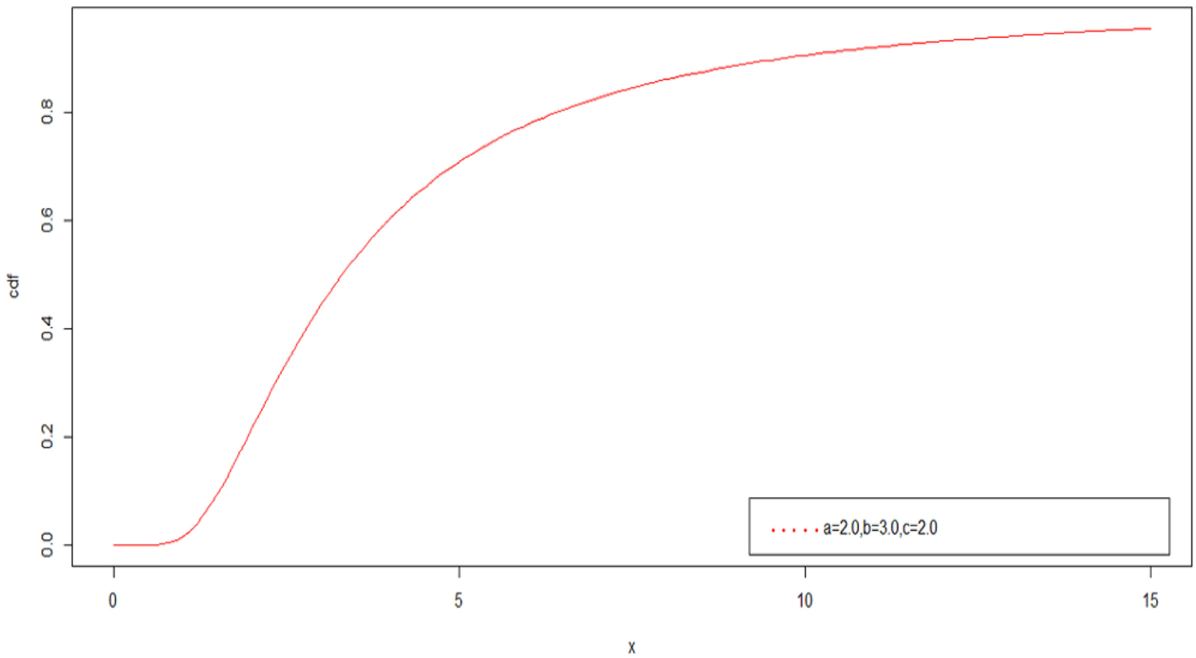

Figure 4.10: CDF of EGIE Distribution at $a = \alpha = 2, b = \beta = 3, c = \theta = 2$

4.6.1 Reliability Analysis for the EGIE Distribution

Expressions for the survival function and the hazard function of the EGIE distribution are provided in this sub-section. Survival function is given as:

$$S(x) = 1 - F(x)$$

Therefore, the expression for the survival function of the Exponentiated Generalised Inverse Exponential distribution is given as:

$$S(x) = 1 - \left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta} \tag{4.39}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

For brevity, a graphical display of the survival function of the EGIE distribution at selected parameter value is as shown in Figure 4.11:

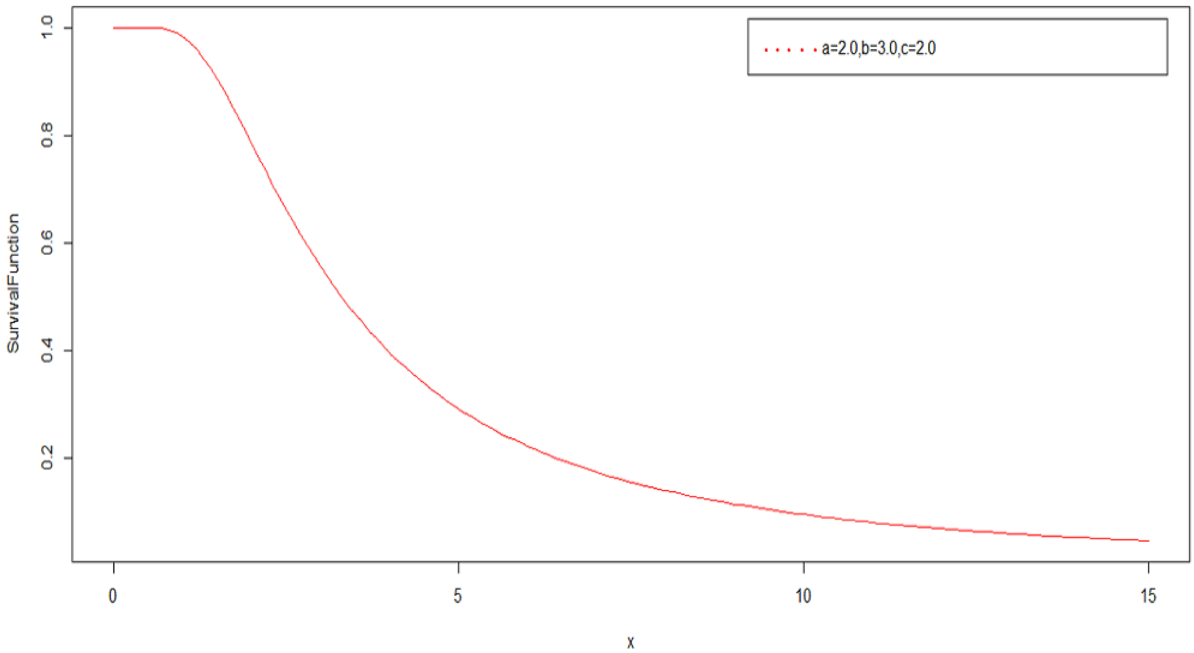

Figure 4.11: Survival Function of EGIE Distribution at $a = \alpha = 2, b = \beta = 3, c = \theta = 2$

Generally, hazard rate is expressed as:

$$h(x) = \frac{f(x)}{1 - F(x)}$$

Therefore, the expression for the hazard function of the Exponentiated Generalised Inverse Exponential distribution is given as:

$$h(x) = \frac{\alpha\beta\theta x^{-2} exp\left(-\frac{\theta}{x}\right)\left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha-1}\left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta-1}}{1 - \left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}} \tag{4.40}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

The graphical representation of the hazard function of the EGIE distribution at some parameter values are given in Figure 4.12. Plots at some other parameter values would produce similar shape(s):

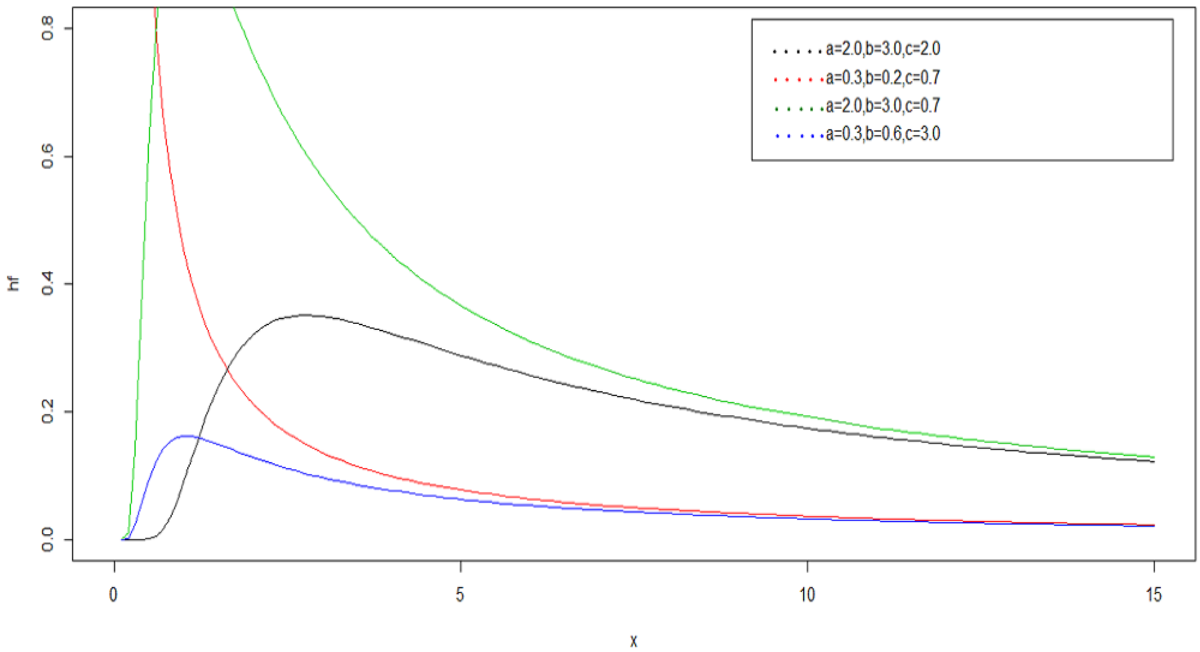

Figure 4.12: Hazard Function of EGIE Distribution where; $a = \alpha, b = \beta, c = \theta$

It can be deduced that the shape of the hazard function for the EGIE distribution could either be inverted bathtub or decreasing depending on the value of the parameters.

4.6.2 Quantile Function and Median of the EGIE Distribution

Quantile function is mathematically given as:

$$Q(u) = F^{-1}(u)$$

With this understanding, the expression for the quantile function of the EGIE distribution can be derived as follows:

$$F(x) = \left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}$$

Let $F(x) = u$, then,

$$u = \left\{1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}$$

$$u^{\frac{1}{\beta}} = 1 - \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}$$

$$1 - u^{\frac{1}{\beta}} = \left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}$$

$$\left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} = 1 - exp\left(-\frac{\theta}{x}\right)$$

$$1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} = exp\left(-\frac{\theta}{x}\right)$$

$$\log\left[1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right] = -\frac{\theta}{x}$$

$$-x\log\left[1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right] = \theta$$

$$x = -\theta\left\{\log\left[1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\right\}^{-1} \tag{4.41}$$

Therefore, random samples can be generated for the EGIE distribution for simulation purposes using the expression in Equation (4.41).

Also, the quantile function of the EGIE distribution is given as:

$$Q(u) = -\theta\left\{\log\left[1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\right\}^{-1} \tag{4.42}$$

Where U has a $Uniform(0,1)$ distribution.

Substituting $u = 0.5$ into Equation (4.43) gives the median of the EGIE distribution as:

$$Median = -\theta\left\{\log\left[1-\left(1-0.5^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\right\}^{-1} \tag{4.43}$$

4.6.3 Distribution of Order Statistics for EGIE Distribution

Let $x_1, x_2, ..., x_n$ denote a random sample from a pdf $f(x)$ and an associated cdf $F(x)$ distributed according to the EGIE distribution, then the pdf of kth order statistics of the EGIE distribution was derived as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}f(x)[F(x)]^{k-1}\left[1-F(x)\right]^{n-k}$$

$$f_{k:n}(x) = n\alpha\beta\theta x^{-2}exp\left(-\frac{\theta}{x}\right)\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha-1}\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta-1}\times$$
$$\left\langle\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}\right\rangle^{k-1}\left\langle 1-\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}\right\rangle^{n-k} \tag{4.44}$$

Therefore, the distribution of minimum order statistics for the EGIE distribution is given as:

$$f_{1:n}(x) = n\alpha\beta\theta x^{-2}exp\left(-\frac{\theta}{x}\right)\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha-1}\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta-1}\times$$
$$\left\langle 1-\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}\right\rangle^{n-1} \tag{4.45}$$

In the same way, the distribution of maximum order statistics for the EGIE distribution is given as:

$$f_{n:n}(x) = n\alpha\beta\theta x^{-2}exp\left(-\frac{\theta}{x}\right)\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha-1}\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta-1}\times$$
$$\left\langle\left\{1-\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\alpha}\right\}^{\beta}\right\rangle^{n-1} \tag{4.46}$$

4.6.4 Estimation of Parameters for the EGIE Distribution

The parameters of the EGIE distribution are estimated through the means of method of maximum likelihood estimation (MLE) as follows; let $X = x_1, x_2, ...x_n$ be a random sample of n independently and identically distributed random variables each having an Exponentiated Generalised Inverse Exponential distribution, the likelihood function is expressed as:

$$L(x_1, x_2, ..., x_n; \alpha, \beta, \theta) = f(x_1, x_2, ..., x_n|\alpha, \beta, \theta) = \prod_{i=1}^{n} f\left(x_i|\alpha, \beta, \theta\right)$$

$$= \prod_{i=1}^{n} \left\langle \alpha\beta\theta x_i^{-2} exp\left(-\frac{\theta}{x_i}\right)\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha-1} \left\{1 - \left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha}\right\}^{\beta-1} \right\rangle$$

Let the log-likelihood be given as $l = \log L(x_1, x_2, ..., x_n; \alpha, \beta, \theta)$.

Therefore,

$$l = n\log(\alpha) + n\log(\beta) + n\log(\theta) - 2\log(x_i) - \sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right) + (\alpha - 1) \times$$
$$\sum_{i=1}^{n} \log\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right] + (\beta - 1)\sum_{i=1}^{n}\log\left\{1 - \left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha}\right\}$$

Following the content of Cordeiro *et al.*, (2013), differentiating l with respect to parameters α, β and θ gives:

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right] + (\beta - 1) \times$$
$$\sum_{i=1}^{n}\frac{\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha}\log\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]}{\left\{1 - \left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha}\right\}} \tag{4.47}$$

$$\frac{dl}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left\{1 - \left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha}\right\} \tag{4.48}$$

$$\frac{dl}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + (\alpha - 1)\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}\right) exp\left(-\frac{\theta}{x_i}\right)}{\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]} + \alpha(\beta - 1) \times$$
$$\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}\right) exp\left(-\frac{\theta}{x_i}\right)\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha-1}}{\left\{1 - \left[1 - exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha}\right\}} \tag{4.49}$$

The solution to the equations of Equations (4.47), (4.48) and (4.49) after being equated to zero gives the maximum likelihood estimates of the parameters α, β and θ. The solution cannot be obtained analytically but it can be gotten with the aid of R software when data sets are available.

4.6.5 Applications of EGIE Distribution to Data sets

In this sub-section, the EGIE distribution was applied to three real life data sets; DATA II, IX and X. The purpose is to assess its flexibility over the GIE distribution and the IE distribution.

For DATA II, the summary has already been provided in Table 4.6.

The performances of the competing distributions are given in Table 4.21:

Table 4.21: Performance Ratings of EGIE distribution Using DATA II

Distribution	Parameters	Log-Likelihood	AIC	Rank
EGIE	$\widehat{\alpha} = 4.6762$ $\widehat{\beta} = 484.5728$ $\widehat{\theta} = 0.4869$	-15.40185	36.8037	1
GIE	$\widehat{\alpha} = 20.766$ $\widehat{\theta} = 6.171$	-17.10457	38.20914	2
IE	$\widehat{\theta} = 1.7247$	-32.66867	67.33734	3

Remark : From Table 4.21, it is clear that the EGIE distribution fits DATA II better than the GIE and IE distributions because it has the highest log-lokelihood value and the lowest AIC value.

For DATA IX, the summary of the data set is given in Table 4.22:

Table 4.22: Summary of data on life of fatigue fracture of Kevlar 373/epoxy

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
76	0.0251	9.0960	1.7360	1.9590	2.4774	1.9406	8.1608

From Table 4.22, it can be seen that the data set is positively skewed with a coefficient of skewness of 1.9406 and a variance of 2.4774.

The performance of the EGIE distribution compared to that of GIE distribution and IE distribution is provided in Table 4.23:

Table 4.23: Performance Ratings of EGIE distribution Using DATA IX

Distribution	Parameters	Log-Likelihood	AIC	Rank
EGIE	$\widehat{\alpha} = 7.595 \times 10^{-1}$ $\widehat{\beta} = 1.446 \times 10^{3}$ $\widehat{\theta} = 5.649 \times 10^{-5}$	-153.5830	313.166	1
GIE	$\widehat{\alpha} = 0.79036$ $\widehat{\theta} = 0.52254$	-161.9691	327.9382	2
IE	$\widehat{\theta} = 0.62487$	-163.1015	328.203	3

Remark : From Table 4.23, it is clear that the EGIE distribution fits DATA IX better than the GIE and IE distributions because it has the highest log-lokelihood value and the lowest AIC value.

For DATA X, the summary of the data set is given in Table 4.24:

Table 4.24: Summary of data on ball bearings

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
23	17.88	173.40	67.80	72.23	1,404.783	0.8811741	3.488857

From Table 4.24, it can be seen that the data set is positively skewed with a coefficient of skewness of 0.8812 and a variance of 1,404.783.

The performance of the EGIE distribution is compared to that of the GIE distribution and IE distribution and the result is as given in Table 4.25:

Table 4.25: Performance Ratings of EGIE distribution Using DATA X

Distribution	**Parameters**	**Log-Likelihood**	**AIC**	**Rank**
EGIE	$\widehat{\alpha} = 2.0621$ $\widehat{\beta} = 27.2256$ $\widehat{\theta} = 10.7126$	-115.3058	236.6116	2
GIE	$\widehat{\alpha} = 5.314$ $\widehat{\theta} = 130.096$	-113.5459	231.0919	1
IE	$\widehat{\theta} = 55.074$	-121.7296	245.4591	3

Remark : From Table 4.25, it is evident that the EGIE distribution failed to perform better than the competing distributions. Instead, the GIE distribution fits DATA X better than the EGIE and IE distributions because it has the highest log-lokelihood value and the lowest AIC value.

4.7 The Weibull Inverse Exponential (WIE) Distribution

Proposition 4 : If a non-negative continuous random variable X is such that; $X \sim WIE(\alpha, \beta, \theta)$, then its cdf and pdf are given as:

$$F(x) = 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1 - exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\} \tag{4.50}$$

and

$$f(x) = \alpha\beta\frac{\theta}{x^2}\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta}}{\left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1 - exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\} \tag{4.51}$$

respectively

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

where α and β are the shape parameters and θ is the scale parameter.

Proof :

For a random variable X, the cdf of the Weibull Inverse Exponential distribution was derived from:

$$F(x) = 1 - exp\left\{-\alpha\left[\frac{G(x)}{1 - G(x)}\right]^{\beta}\right\} \tag{4.52}$$

where $\alpha > 0, \beta > 0$ are additional shape parameters.

In this context, $G(x)$ and $g(x)$ are the cdf and pdf of the Inverse Exponential distribution as given in Equations (3.1) and (3.2) respectively.

That is,

$$G(x) = exp\left(-\frac{\theta}{x}\right)$$

and

$$g(x) = \frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)$$

for $x > 0, \theta > 0$

Now, inserting Equation (3.1) into Equation (4.52) gives the cdf of the Weibull Inverse Exponential (WIE) distribution as:

$$F(x) = 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

Its corresponding pdf was derived from:

$$f(x) = \alpha\beta g(x)\frac{[G(x)]^{\beta-1}}{[1-G(x)]^{\beta+1}}exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right\} \tag{4.53}$$

Inserting Equations (3.1) and (3.2) into Equation (4.53) gives the pdf of the Weibull Inverse Exponential distribution as:

$$f(x) = \alpha\beta\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$$

which was simplified to give:

$$f(x) = \alpha\beta\frac{\theta}{x^2}\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

Corollary 4 : The Weibull Inverse Exponential distribution is a valid distribution. It follows that:

$$\lim_{x\to\delta} F(x) = 1$$

where $\delta \to \infty$.

Proof :

$$\lim_{x\to\delta} F(x) = \lim_{x\to\delta}\left\langle 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle$$

$$= 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{\delta}\right)}{1-exp\left(-\frac{\theta}{\delta}\right)}\right]^{\beta}\right\}$$

where $exp\left(-\frac{\theta}{\delta}\right) \simeq 1$, since $\delta \to \infty$.

Therefore,

$$\lim_{x \to \delta} F(x) = \left\{1 - exp\left[-\alpha\left(\infty\right)^{\beta}\right]\right\}$$

$$= 1 - 0$$

$$= 1$$

This completes the proof.

The graphical representation of the pdf of the WIE distribution at various selected parameter values are given in Figure 4.13. Plots at other parameter values would produce similar shape(s):

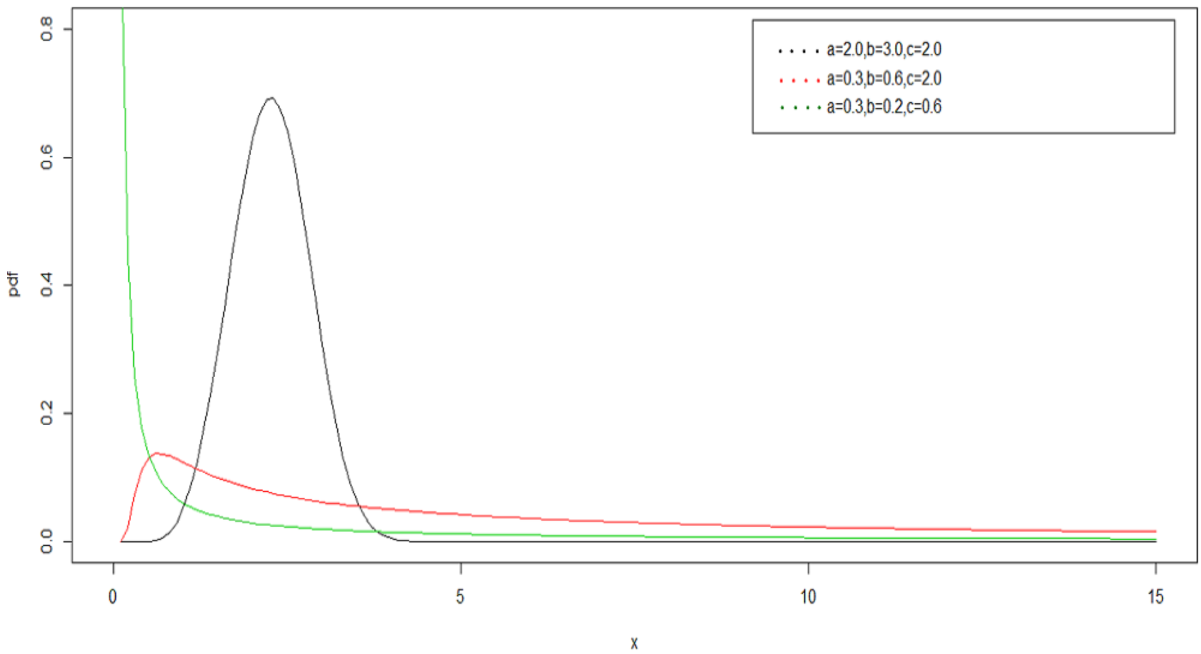

Figure 4.13: PDF of WIE Distribution where; $a = \alpha, b = \beta, c = \theta$

From Figure 4.13, it can be seen that the shape of the WIE distribution is either unimodal (inverted bathtub) or decreasing (depending on the parameter values).

For brevity, a graphical display of the cdf of the WIE distribution is shown in Figure 4.14:

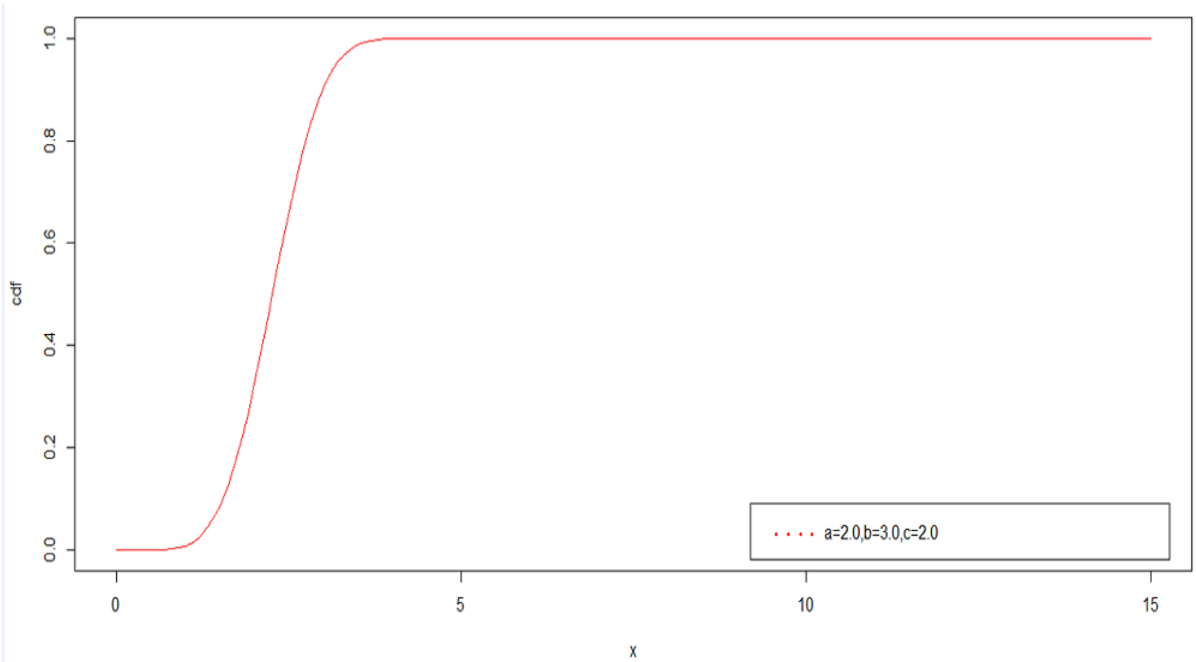

Figure 4.14: CDF of WIE Distribution at $a = \alpha = 2, b = \beta = 3, c = \theta = 2$

4.7.1 Reliability Analysis for the Weibull Inverse Exponential Distribution

In this section, the expressions for the survival function and hazard function of the Weibull Inverse Exponential distribution are derived.

Survival function is given as:

$$S(x) = 1 - F(x)$$

In this context, $F(x)$ is the cdf of the WIE distribution. Therefore, the expression for the survival function of the WIE distribution was derived as follows:

$$S(x) = 1 - \left\langle 1 - exp\left\{ -\alpha \left[\frac{exp\left(-\frac{\theta}{x}\right)}{1 - exp\left(-\frac{\theta}{x}\right)} \right]^{\beta} \right\} \right\rangle$$

Thus:

$$S(x) = exp\left\{ -\alpha \left[\frac{exp\left(-\frac{\theta}{x}\right)}{1 - exp\left(-\frac{\theta}{x}\right)} \right]^{\beta} \right\} \tag{4.54}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

For brevity, a graphical display of the survival function of the WIE distribution at selected parameter value is shown in Figure 4.15:

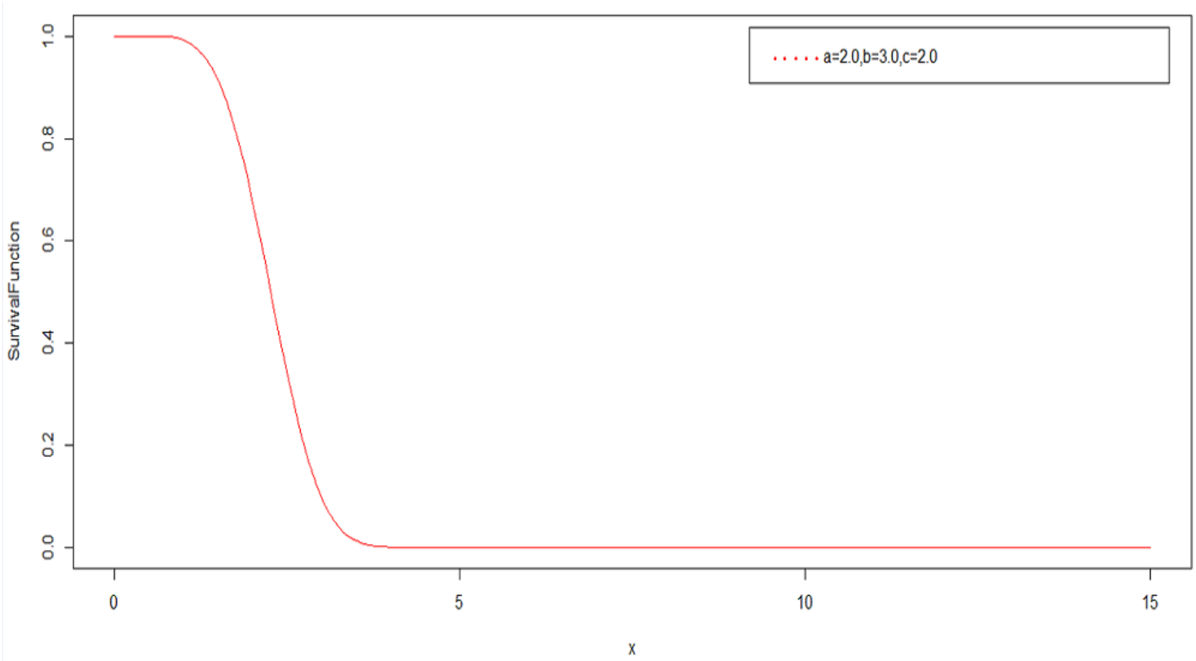

Figure 4.15: Survival Function of WIE Distribution at $a = \alpha = 2, b = \beta = 3, c = \theta = 2$

Hazard function is given as:

$$h(x) = \frac{f(x)}{1 - F(x)}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the WIE distribution respectively.

Then, the hazard function of the WIE distribution was derived as:

$$h(x) = \frac{\alpha\beta\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}}{1 - \left\langle 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle}$$

$$h(x) = \frac{\alpha\beta\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}}{exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}}$$

$$h(x) = \alpha\beta\frac{\theta}{x^2}\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta}}{\left[1 - exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}} \tag{4.55}$$

for $x > 0, \alpha > 0, \beta > 0, \theta > 0$

The graphical representation of the hazard function of the WIE distribution at varied parameter values are shown in Figure 4.16. Plots at other parameter values would produce similar shape(s):

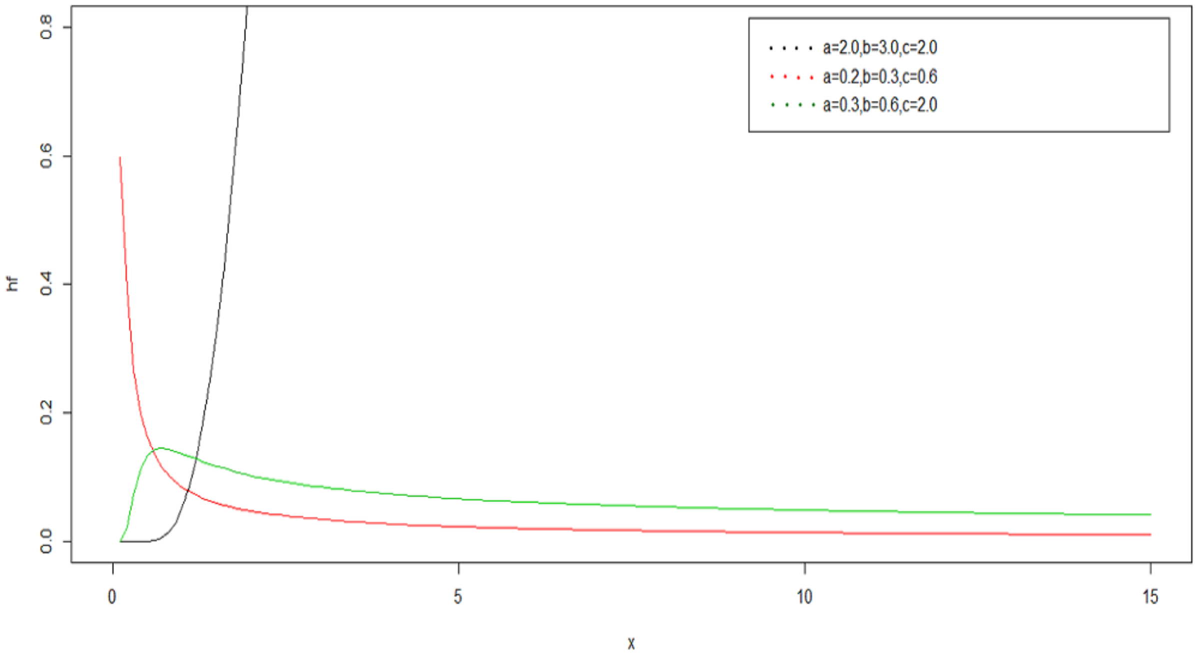

Figure 4.16: Hazard Function of WIE Distribution where; $a = \alpha, b = \beta, c = \theta$

Remark : From the plots displayed, it can be deduced that the shape of the hazard function of the Weibull Inverse Exponential distribution could either be unimodal or decreasing (depending on the values of the parameters).

4.7.2 Quantile Function and Median for the WIE Distribution

The mathematical expression for the quantile function is:

$$Q(u) = F^{-1}(u)$$

Therefore, the quantile function of the WIE distribution was derived from:

$$F(x) = 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$$

Let $u = F(x)$, then;

$$u = 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$$

$$1 - u = exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$$

$$\log(1-u) = -\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}$$

$$-\alpha^{-1}\log(1-u) = \left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}$$

$$\left[-\alpha^{-1}\log(1-u)\right]^{\frac{1}{\beta}} = \left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]$$

Divide both the numerator and denomerator of the R.H.S by $exp(-\frac{\theta}{x})$:

$$\left[-\alpha^{-1}\log(1-u)\right]^{\frac{1}{\beta}} = \left[\frac{1}{\frac{1}{exp(-\frac{\theta}{x})} - 1}\right]$$

$$\left[-\alpha^{-1}\log(1-u)\right]^{\frac{1}{\beta}} \times \left[\frac{1}{exp(-\frac{\theta}{x})} - 1\right] = 1$$

Divide both sides by $[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}$:

$$\frac{1}{exp(-\frac{\theta}{x})} - 1 = \frac{1}{[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}}$$

$$\frac{1}{exp(-\frac{\theta}{x})} = \frac{1}{[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}} + 1$$

$$exp(\frac{\theta}{x}) = \frac{1}{[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}} + 1$$

Take the natural logarithm of both sides:

$$\log\left[exp\left(\frac{\theta}{x}\right)\right] = \log\left\{\frac{1}{[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}} + 1\right\}$$

$$\frac{\theta}{x} = \log\left\{\frac{1}{[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}} + 1\right\}$$

$$x = \frac{\theta}{\log\left\{\frac{1}{[-\alpha^{-1}\log(1-u)]^{\frac{1}{\beta}}} + 1\right\}}$$

Therefore,

$$x = \frac{\theta}{\log\left\{[-\alpha^{-1}\log(1-u)]^{-\frac{1}{\beta}} + 1\right\}} \tag{4.56}$$

This means that random samples can be simulated from the WIE distribution using the expression in Equation (4.56).

Equivalently, the quantile function of the WIE distribution is given as:

$$Q(u) = \frac{\theta}{\log\left\{[-\alpha^{-1}\log(1-u)]^{-\frac{1}{\beta}} + 1\right\}} \tag{4.57}$$

where $U \sim Uniform(0,1)$

The median of the WIE distribution can be derived by substituting $u = 0.5$ in Equation (4.57) to give:

$$Median = \frac{\theta}{\log\left\{[-\alpha^{-1}\log(0.5)]^{-\frac{1}{\beta}} + 1\right\}} \tag{4.58}$$

4.7.3 Distribution of Order Statistics for WIE Distribution

Let $x_1, x_2, ..., x_n$ denote a random sample from a pdf $f(x)$ and an associated cdf $F(x)$ distributed according to the WIE distribution, then the pdf of kth order statistics of the WIE distribution was derived as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}$$

$$f_{k:n}(x) = n\alpha\beta\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\times$$
$$\left\langle 1-exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle^{k-1}\times\left\langle exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle^{n-k} \quad (4.59)$$

Therefore, the distribution of minimum order statistics for the WIE distribution is given as:

$$f_{1:n}(x) = n\alpha\beta\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\times$$
$$\left\langle exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle^{n-1} \quad (4.60)$$

In the same way, the distribution of maximum order statistics for the WIE distribution is:

$$f_{n:n}(x) = n\alpha\beta\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\times$$
$$\left\langle 1-exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle^{n-1} \quad (4.61)$$

4.7.4 Estimation of Parameters for the WIE Distribution

The parameters of the WIE distribution were estimated through the means of method of maximum likelihood estimation (MLE) as follows: let $X = x_1, x_2, ..., x_n$ be a random sample of n independently and identically distributed random variables each having an Weibull Inverse Exponential distribution, the likelihood function is expressed as:

$$L(x_1, x_2, ..., x_n; \alpha, \beta, \theta) = f(x_1, x_2, ..., x_n|\alpha, \beta, \theta) = \prod_{i=1}^{n} f\left(x_i|\alpha, \beta, \theta\right)$$

$$= \prod_{i=1}^{n}\left\langle \alpha\beta\frac{\theta}{x_i^2}exp\left(-\frac{\theta}{x_i}\right)\frac{\left[exp\left(-\frac{\theta}{x_i}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x_i}\right)\right]^{\beta+1}}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}\right\}\right\rangle$$

$$= \prod_{i=1}^{n} \left\langle \alpha\beta \frac{\theta}{x_i^2} \frac{exp\left(-\frac{\theta}{x_i}\right)\left[exp\left(-\frac{\theta}{x_i}\right)\right]^{\beta-1}}{\left[1-exp\left(-\frac{\theta}{x_i}\right)\right]^{\beta+1}} exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}\right\}\right\rangle$$

$$= \prod_{i=1}^{n} \left\langle \alpha\beta \frac{\theta}{x_i^2} \frac{\left[exp\left(-\frac{\theta}{x_i}\right)\right]^{\beta}}{\left[1-exp\left(-\frac{\theta}{x_i}\right)\right]^{\beta+1}} exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}\right\}\right\rangle$$

$$= \alpha^n\beta^n\theta^n \prod_{i=1}^{n}\left(\frac{1}{x_i^2}\right)\left\langle \frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{\beta}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}\right\rangle^{n}\left\langle exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1-exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right\rangle^{n}$$

$$= \alpha^n\beta^n\theta^n \prod_{i=1}^{n}\left(\frac{1}{x_i^2}\right)\frac{\left[exp\left(-\frac{\theta}{x}\right)\right]^{n\beta}}{\left[1-exp\left(-\frac{\theta}{x}\right)\right]^{n(\beta+1)}} exp\left\{-\alpha\sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}\right\}$$

Let $l = \log L(x_1, x_2, ..., x_n; \alpha, \beta, \theta)$

Then, the log-likelihood function denoted by l is given as:

$$l = n\log(\alpha) + n\log(\beta) + n\log(\theta) + \sum_{i=1}^{n}\log\left(\frac{1}{x_i^2}\right) + \beta\sum_{i=1}^{n}\left(-\frac{\theta}{x_i}\right) - (\beta+1)\times$$

$$\sum_{i=1}^{n}\log\left[1-exp\left(-\frac{\theta}{x_i}\right)\right] - \alpha\sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}$$

Thus:

$$l = n\log(\alpha) + n\log(\beta) + n\log(\theta) - 2\sum_{i=1}^{n}\log(x_i) + \beta\sum_{i=1}^{n}\left(-\frac{\theta}{x_i}\right) -$$

$$(\beta+1)\sum_{i=1}^{n}\log\left[1-exp\left(-\frac{\theta}{x_i}\right)\right] - \alpha\sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta} \quad (4.62)$$

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}$$

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right) - \sum_{i=1}^{n}\log\left[1-exp\left(-\frac{\theta}{x_i}\right)\right] - \alpha\sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}\times$$

$$\log\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]$$

$$\frac{dl}{d\theta} = \frac{n}{\theta} - \beta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) - (\beta+1)\sum_{i=1}^{n}\frac{\frac{1}{x_i}exp\left(-\frac{\theta}{x_i}\right)}{\left[1-exp\left(-\frac{\theta}{x_i}\right)\right]} - \alpha\beta\sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta-1}\times$$

$H(x;\theta)$

where: $H(x;\theta) = \frac{d}{d\theta}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1-exp\left(-\frac{\theta}{x_i}\right)}\right]$

Solving the resulting equations of $\frac{dl}{d\alpha} = 0$, $\frac{dl}{d\beta} = 0$ and $\frac{dl}{d\theta} = 0$ gives the maximum likelihood estimates of parameters α, β and θ. Though, the solution cannot be obtained analytically but it can obtain numerically with the aid of statistical software.

4.7.5 Application of WIE Distribution to Data set

In this study, the flexibility of the WIE distribution was assessed using DATA XI and VI.

For DATA XI, the summary of the data set is as shown in Table 4.26:

Table 4.26: Summary of data on bladder cancer patients

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
128	0.080	79.050	6.395	9.366	110.425	3.286569	18.48308

From Table 4.26, it can be seen that the data set is positively skewed with a coefficient of skewness value of 3.2866 and variance of 110.425.

The performance of the WIE distribution over that of IE distribution was assessed and the result is given in Table 4.27:

Table 4.27: Performance Ratings of WIE distribution Using DATA XI

Distribution	Parameters	Log-Likelihood	AIC	Rank
WIE	$\widehat{\alpha} = 0.398933$ $\widehat{\beta} = 0.310546$ $\widehat{\theta} = 0.004703$	-227.9772	461.9544	1
IE	$\widehat{\theta} = 2.4847$	-460.3823	922.7646	2

Remark : From Table 4.27, it is evident that the WIE distribution fits DATA XI better than the IE distribution because it has the highest log-likelihood value and lowest AIC value.

DATA VI which is considered to be over-dispersed in this study is also considered and the summary of the data set has already been given in Table 4.14.

The performance of the WIE distribution with respect to DATA VI is as given in Table 4.28:

Table 4.28: Performance Ratings of WIE distribution Using DATA VI

Distribution	Parameters	Log-Likelihood	AIC	Rank
WIE	$\widehat{\alpha} = 0.7609$ $\widehat{\beta} = 0.2488$ $\widehat{\theta} = 0.0018$	-200.4028	406.8056	1
IE	$\widehat{\theta} = 76.7000$	-279.5773	561.1546	2

Remark : The result in Table 4.28 shows that the WIE distribution can fit DATA VI better than the IE distribution. Meanwhile, this is the first time in all the analyses provided in this study that a compound distribution would perform better than its sub-models when applied to a data set that has a very large variance. Though, the standard error posed by the scale parameter was unobtainable from the results gotten in R software.

4.8 The Exponential Generalised Family of Distributions

Following the concept of Bourguignon *et al.*, (2014), another generalisation named the Exponential-G family of distributions was derived. This was obtained using the relation:

$$\Phi(x) = \int_0^{\frac{\Xi(x)}{1-\Xi(x)}} \alpha e^{-\alpha t} dt \tag{4.63}$$

where $\Xi(x)$ is the cdf of any arbitrary distribution.

It then follows that:

$$\Phi(x) = \left[-\frac{\alpha e^{-\alpha t}}{\alpha} \right]_0^{\frac{\Xi(x)}{1-\Xi(x)}}$$

$$\Phi(x) = \left[-e^{-\alpha t} \right]_0^{\frac{\Xi(x)}{1-\Xi(x)}}$$

$$= \left[-e^{-\alpha\left(\frac{\Xi(x)}{1-\Xi(x)}\right)} + 1 \right]$$

$$\Phi(x) = 1 - exp\left\{ -\alpha \left[\frac{\Xi(x)}{1-\Xi(x)} \right] \right\} \tag{4.64}$$

To derive the corresponding pdf, $\Phi(x)$ was differentiated with respect to x to give:

$$\frac{d\Phi(x)}{dx} = \phi(x) = \alpha \frac{\xi(x)}{[1-\Xi(x)]^2} exp\left\{ -\alpha \left[\frac{\Xi(x)}{1-\Xi(x)} \right] \right\} \tag{4.65}$$

where $\alpha > 0$ is a shape parameter whose role is to induce skewness and vary tail weights.

$\xi(x) = \frac{d\Phi(x)}{dx}$ is the pdf of any arbitrary distribution

These results are similar to the result of Bourguignon *et al.*, (2014) when $\beta = 1$.

Proposition 5 : Let X denote a non-negative continuous random variable and let $\Xi(x)$ be a continuous baseline cdf which exists, then the cdf of the Exponential Generalised family of distributions with only one extra parameter is given as:

$$\Phi(x) = 1 - exp\left\{ -\alpha \left[\frac{\Xi(x)}{1-\Xi(x)} \right] \right\} \tag{4.66}$$

for $x > 0, \alpha > 0$

Its associated pdf is given as:

$$\phi(x) = \alpha \frac{\xi(x)}{[1-\Xi(x)]^2} exp\left\{-\alpha \left[\frac{\Xi(x)}{1-\Xi(x)}\right]\right\} \tag{4.67}$$

for $x > 0, \alpha > 0$

4.8.1 Some Exponential Generalised Models

In this section, some popular standard distributions were generalised using the Exponential Generalised family of distributions.

Exponential Weibull Distribution

Consider the Weibull distribution with cdf and pdf given as:

$$\Xi(x) = 1 - e^{-\lambda x^{\beta}}$$

and

$$\xi(x) = \lambda\beta x^{\beta-1} e^{-\lambda x^{\beta}}$$

respectively

for $x > 0, \lambda > 0, \beta > 0$

where λ is a shape parameter and β is a scale parameter.

Then, the cdf of the Exponential Weibull distribution is given as:

$$\Phi(x) = 1 - exp\left\{-\alpha \left[\frac{1-e^{-\lambda x^{\beta}}}{e^{-\lambda x^{\beta}}}\right]\right\} \tag{4.68}$$

for $x > 0, \alpha > 0, \lambda > 0, \beta > 0$

The corresponding pdf is given as:

$$\phi(x) = \frac{\alpha\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}}{\left[e^{-\lambda x^{\beta}}\right]^{2}} exp\left\{-\alpha\left[\frac{1-e^{-\lambda x^{\beta}}}{e^{-\lambda x^{\beta}}}\right]\right\} \tag{4.69}$$

for $x > 0, \alpha > 0, \lambda > 0, \beta > 0$

Exponential Pareto Distribution

Consider the Pareto distribution with cdf and pdf given as:

$$\Xi(x) = 1 - \left(\frac{k}{x}\right)^{\theta}$$

and

$$\xi(x) = \frac{\theta k^{\theta}}{x^{\theta+1}}$$

respectively

for $x \geq k, \theta > 0$

where k is a scale parameter and θ is a shape parameter.

Then, the cdf of the Exponential Pareto distribution is given as:

$$\Phi(x) = 1 - exp\left\{-\alpha\left[\frac{1-\left(\frac{k}{x}\right)^{\theta}}{\left(\frac{k}{x}\right)^{\theta}}\right]\right\} \tag{4.70}$$

for $x \geq k, \alpha > 0, \theta > 0$

The corresponding pdf is given as:

$$\phi(x) = \frac{\alpha\theta k^{\theta}}{x^{\theta+1}} \frac{1}{\left[\left(\frac{k}{x}\right)^{\theta}\right]^{2}} exp\left\{-\alpha\left[\frac{1-\left(\frac{k}{x}\right)^{\theta}}{\left(\frac{k}{x}\right)^{\theta}}\right]\right\} \tag{4.71}$$

for $x \geq k, \alpha > 0, \theta > 0$

Exponential Kumaraswamy Distribution

Consider the Kumaraswamy distribution with cdf and pdf given as:

$$\Xi(x) = \left[1 - (1 - x^a)^b\right]$$

and

$$\xi(x) = abx^{a-1}(1 - x^a)^{b-1}$$

respectively

for $x \in [0, 1], a > 0, b > 0$

Then, the cdf of the Exponential Kumaraswamy distribution is given as:

$$\Phi(x) = 1 - exp\left\{-\alpha\left[\frac{1 - (1 - x^a)^b}{(1 - x^a)^b}\right]\right\} \tag{4.72}$$

for $x > 0, \alpha > 0, a > 0, b > 0$

The corresponding pdf is given as:

$$\phi(x) = ab\alpha x^{a-1}(1 - x^a)^{b-1}\frac{1}{\left[(1 - x^a)^b\right]^2}exp\left\{-\alpha\left[\frac{1 - (1 - x^a)^b}{(1 - x^a)^b}\right]\right\} \tag{4.73}$$

for $x > 0, \alpha > 0, a > 0, b > 0$

Exponential Inverse Exponential Distribution

The cdf and pdf of the Exponential Inverse Exponential distribution are given as:

$$\Phi(x) = 1 - exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1 - exp\left(-\frac{\theta}{x}\right)}\right]\right\} \tag{4.74}$$

and

$$\phi(x) = \alpha\frac{\theta}{x^2}exp\left(-\frac{\theta}{x}\right)\frac{1}{\left[1 - exp\left(-\frac{\theta}{x}\right)\right]^2}exp\left\{-\alpha\left[\frac{exp\left(-\frac{\theta}{x}\right)}{1 - exp\left(-\frac{\theta}{x}\right)}\right]\right\} \tag{4.75}$$

respectively

for $x > 0, \alpha > 0, \theta > 0$

where α is the shape parameter and θ is the scale parameter.

4.8.2 Statistical Properties of the Exponential Generalised Family of Distributions

In this section, some basic properties of the Exponential Generalised family of distributions are discussed.

Reliability Analysis

The survival function for the Exponential family of distributions was derived from:

$$S(x) = 1 - \Phi(x)$$

$$S(x) = 1 - \left\langle 1 - exp\left\{-\alpha\left[\frac{\Xi(x)}{1-\Xi(x)}\right\rangle\right\}\right]$$

Therefore,

$$S(x) = exp\left\{-\alpha\left[\frac{\Xi(x)}{1-\Xi(x)}\right]\right\} \tag{4.76}$$

for $x > 0, \alpha > 0$

Meanwhile, its failure rate was derived from:

$$h(x) = \frac{\phi(x)}{1-\Phi(x)}$$

This becomes:

$$h(x) = \frac{\alpha\xi(x)\frac{1}{[1-\Xi(x)]^2} exp\left\{-\alpha\left[\frac{\Xi(x)}{1-\Xi(x)}\right]\right\}}{exp\left\{-\alpha\left[\frac{\Xi(x)}{1-\Xi(x)}\right]\right\}}$$

Therefore,

$$h(x) = \alpha\xi(x)\frac{1}{[1-\Xi(x)]^2} \tag{4.77}$$

for $x > 0, \alpha > 0$

where $\xi(x)$ is the pdf of the parent distribution.

Parameter Estimation

The unknown parameters of the Exponential-G distribution can be estimated using the method of maximum likelihood. Let $x_1, x_2, ..., x_n$ be a random sample each having the pdf of the Exponential-G distribution, then the likelihood function is given as:

$$\phi(x_1, x_2, ...x_n) = \prod_{i=1}^{n}\left\langle \alpha\xi(x_i)\frac{1}{[1-\Xi(x_i)]^2} exp\left\{-\alpha\left[\frac{\Xi(x_i)}{1-\Xi(x_i)}\right]\right\}\right\rangle$$

The log-likelihood function is given as:

$$l = n\log(\alpha) + \sum_{i=1}^{n}\log[\xi(x_i)] - 2\sum_{i=1}^{n}\log[1-\Xi(x_i)] - \alpha\sum_{i=1}^{n}[W(x_i)] \tag{4.78}$$

where $W(x_i) = \frac{\Xi(x_i)}{1-\Xi(x_i)}$.

The maximum likelihood estimates can be obtained from the expression in Equation (4.78). Statistical software can be used to solve the resulting non-linear equations numerically because the equations cannot be solved analytically.

4.8.3 Applications Involving the Exponential Generalised Family of Distributions

As an example, the Exponential Inverse Exponential distribution was applied to three real life data sets and its performance over the Inverse Exponential distribution was compared. For these analyses, DATA VI, XI and XII are used.

The pdf of the Exponential Inverse Exponential distribution has been given in Equation (4.75). Hence, its log-likelihood function can be given as:

$$l = n\log(\theta) + n\log(\alpha) - 2\sum_{i=1}^{n}\log(x_i) - \theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)$$
$$-2\sum_{i=1}^{n}\log\left[1 - exp\left(-\frac{\theta}{x_i}\right)\right] - \alpha\sum_{i=1}^{n}\left[\frac{exp\left(-\frac{\theta}{x_i}\right)}{1 - exp\left(-\frac{\theta}{x_i}\right)}\right] \tag{4.79}$$

For DATA VI, the summary of the data set has already been provided in Table 4.14.

At first, the performance of the EIE distribution was compared with that of IE distribution and the result is as shown in Table 4.29:

Table 4.29: Performance Ratings of EIE distribution Using DATA VI

Distribution	**Parameters**	**Log-Likelihood**	**AIC**	**Rank**
EIE	$\widehat{\theta} = 33.44690$ $\widehat{\alpha} = 0.16085$	-280.4043	564.8086	2
IE	$\widehat{\theta} = 76.7000$	-279.5773	561.1546	1

Remark : From the result in Table 4.29 using the data set that is considered to be over-dispersed, the EIE distribution did not perform better than the IE distribution in fitting DATA VI.

For DATA XI, the summary of the data set has been provided in Table 4.26.

The performance of the EIE distribution when applied to DATA XI is as given in Table 4.30:

Table 4.30: Performance Ratings of EIE distribution Using DATA XI

Distribution	Parameters	Log-Likelihood	AIC	Rank
EIE	$\widehat{\theta} = 0.21092$ $\widehat{\alpha} = 0.02277$	-413.6499	831.2998	1
IE	$\widehat{\theta} = 2.4847$	-460.3823	922.7646	2

Remark : From the result in Table 4.30, it can be seen that the EIE distribution fits DATA XI better than the IE distribution because it has the highest log-likelihood value and the lowest AIC value.

Lastly, the EIE distribution was also applied to DATA XII and the summary of the data set is given in Table 4.31:

Table 4.31: Summary of data on breast cancer

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
121	0.30	154.00	40.00	46.33	1,244.464	1.04318	3.402139

From Table 4.31, it can be seen that the data set on breast cancer is positively skewed with a coefficient of skewness of 1.04318 and variance of 1,244.464. This variance is also large but it is not as large as that of DATA VI.

The performance of the EIE distribution with respect to DATA XII is provided in Table 4.32:

Table 4.32: Performance Ratings of TIE distribution Using DATA XII

Distribution	Parameters	Log-Likelihood	AIC	Rank
EIE	$\widehat{\theta} = 0.350733$ $\widehat{\alpha} = 0.007599$	-584.9013	1,173.803	1
IE	$\widehat{\theta} = 10.3215$	-677.2791	1,356.558	2

Remark : From the result in Table 4.32, it can be seen that the EIE distribution fits DATA XII better than the IE distribution because it has the highest log-likelihood value and the lowest AIC value.

4.9 Simulation Study

A data set of size $m = 1,000$ was simulated from a well-known two parameter lifetime distribution (Weibull distribution) with parameters $\alpha = 2, \theta = 3$. Six different data sets with sizes $n = 50, 100, 150, 200, 500$ and 800 were extracted from the main data set (m) using the method of simple random sampling with the aid of R software. Then, the two-parameter Exponentiated Inverse Exponential (EIE) distribution was fitted to the data sets. The idea is to know how the parameters of the EIE distribution would behave. The maxLik package in R software was used to estimate the maximum likelihood estimates of parameters α and θ and the algorithm used to obtain the result is given in Appendix E. However, this algorithm can be modified to simulate for the KIE distribution, TIE distribution, EGIE distribution and WIE distribution. The result is as shown in Table 4.33:

Table 4.33: Result from the simulated data

n	Parameter	MLE	Standard Error	MSE	Bias	Abs. Bias
50	$\alpha = 2$	1.5592	0.6067	0.012134	0.4408	0.4408
	$\theta = 3$	2.4548	0.6036	0.012072	0.5452	0.5452
100	$\alpha = 2$	1.0018	0.2855	0.002855	0.9982	0.9982
	$\theta = 3$	1.7954	0.3573	0.003573	1.2046	1.2046
150	$\alpha = 2$	2.4363	0.5033	0.003355	-0.4363	0.4363
	$\theta = 3$	3.4207	0.3987	0.002658	-0.4207	0.4207
200	$\alpha = 2$	2.0845	0.3905	0.001953	-0.0845	0.0845
	$\theta = 3$	3.2450	0.3572	0.001786	-0.2450	0.2450
500	$\alpha = 2$	1.4384	0.1812	0.000362	0.5616	0.5616
	$\theta = 3$	2.4072	0.1952	0.000390	0.5928	0.5928
800	$\alpha = 2$	1.6760	0.1616	0.000202	0.3240	0.3240
	$\theta = 3$	2.6778	0.1603	0.000200	0.3222	0.3222

Remark : It can be seen that the error of parameters α and θ decreases as the sample size (n) increases but there is an exception for parameter α when $n = 150$. It can also be seen that the biasedness of the parameters are not large (that is, the parameter estimates are close to the true parameter values). The biasedness in this case cannot be zero because the data set was simulated from Weibull distribution and Exponential Inverse Exponential distribution was fitted on it. It is expected that there will be a level of biasedness.

4.10 Summary

In this chapter, all the results have been successfully obtained to capture all the stated objectives. The statistical table for the Inverse Exponential distribution, densities for all the derived compound distributions, plots, statistical properties, estimation of model parameters and simulation study have all been provided.

The pdf and cdf of the KIE, TIE, EGIE and WIE distributions have been successfully derived. The distributions have three parameters each; two shape parameters and a scale parameter except for the TIE distribution which has just two parameters; a transmuted parameter and a scale parameter. Plots for the pdf and cdf of these distributions have been provided. All these four distributions are positively skewed; this is evident through the plots provided. The shapes of the four distributions are respectively unimodal, except in the following cases:

i When the value of each of the three parameters of the KIE distribution is less than one, the shape of the distribution decreases as displayed in Figure 4.1.

ii When the value of each of the two parameters of the TIE distribution is less than one with the transmuted parameter being positive, the shape of the distribution decreases as displayed in Figure 4.5.

iii When the value of each of the three parameters of the EGIE distribution is less than one, the shape of the distribution decreases as displayed in Figure 4.9.

iv When the value of each of the three parameters of the WIE distribution is less than one, the shape of the distribution decreases as displayed in Figure 4.13.

Some basic statistical properties have been identified with explicit expressions for the survival function, hazard function and quantile function of the KIE, TIE, EGIE and WIE distributions derived. The shape of the hazard function of the four compound distributions could either be unimodal or decreasing depending on the value of the parameters. In addition, the shape of the WIE distribution could be increasing. This implies that the distributions can be used to model real life phenomena with inverted bathtub failure rates and decreasing failure rates. In addition to that, the WIE distribution can also model real life phenomena with increasing failure rates.

The distributions of order statistics (including the minimum and maximum) for the four derived compound distributions have been successfully obtained while the method of maximum likelihood estimation was used in estimating the parameters of the distributions.

The KIE distribution was applied to six real life data sets; DATA I to DATA VI, and its performance was compared with that of the Generalised Inverse Exponential (GIE) distribution and Inverse Exponential (IE) distribution. Based on the results of the analysis, the KIE distribution is indeed an improvement over the two other competing distributions as it performs better based on the value of its log-likelihood and AIC. On the other hand, when the variance of the data set is far more than its mean, that is, when the data set is over-dispersed (which might be due to the presence of outliers in the data set), the KIE distribution tends to perform poorly. This can be seen in Tables 4.11, 4.13 and 4.15.

The same thing is applicable to the TIE and EGIE distributions. Although, the TIE distribution was applied to DATA VI, VII and VIII while EGIE distribution was applied to DATA II, IX and XI but their inabilities to properly fit data sets with high variance have been seen in Tables 4.20 and 4.25 respectively. The WIE distribution on the other hand was applied to data sets with both low and high variances and it performs better than the IE distribution (though the standard error for the scale parameter was unobtainable from the software).

A new family of generalised distribution named the Exponential Generalised family of distributions has also been defined and explored. As a result, the Exponential Weibull distribution, Exponential Pareto distribution, Exponential Kumaraswamy distribution and Exponential Inverse Exponential distribution have been derived. The flexibility of the Exponential Generalised family of distributions was assessed and investigated using the Exponential Inverse Exponential distribution, it was confirmed that the distribution performs better than its sub-model; Inverse Exponential distribution.

One of the advantages of the Exponential Generalised family of distribution over some other families is its tractability; it has just only one additional parameter. This means that when such family of distribution is used, the resulting compound distribution would not have too many parameters. This would make computation easier and would not lead to much mathematical complexities as in the case of Beta generated distributions.

A simulation study was conducted using the Weibull distribution which is always regarded as being versatile and whose application is widely known in modelling lifetime data sets. A data set of size $m = 1,000$ was generated and random samples of size $n = 50, 100, 150, 200, 500$ and 800 were subsequently drawn. The Exponential Inverse Exponential (EIE) distribution was fitted to the data sets and the maximum likelihood estimates of its parameters including the associated standard errors were obtained using the maxLik package in R software. It was discovered that as the sample size increases, the standard errors of the parameters decreases, also the estimated parameters are close to the true parameter values, hence, the bias generated as a result of using EIE distribution instead of Weibull distribution is not large. The next chapter includes the overview, conclusions and recommendations for this study.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter, the conclusions and recommendations resulting from this study are presented, directions for further studies are also identified, the basic facts that this study contributes to knowledge are clearly stated and followed by a summary of the study.

5.2 Conclusion

The statistical table for the Inverse Exponential distribution is new (to the best of the researcher's knowledge) and the method used in this study could be modified to generate statistical tables for other distributions. As mentioned in previous Chapters; very little has been done on the generalised version of the Inverse Exponential distribution (Dey and Pradhan, 2014) and the Inverse Exponential distribution is deficient in ability to fit data sets that are highly skewed (Abouammoh and Alshingiti, 2009). This study has been able to generalise the Inverse Exponential distribution successfully with four different generalised families of distributions. Besides, skewness has been induced into the Inverse Exponential distribution by the additional shape parameter(s) contained in the generalised families of distributions used.

The derived compound distributions are positively skewed and their shapes have been successfully established; these include unimodality and decreasing. The cdfs of all the derived compound distributions equal one when the value of x approaches infinity, this means that all these cdfs are valid and by extension, their corresponding pdfs are also valid.

The shapes of the derived compound distributions have also been established, these include; unimodality, decreasing and increasing. It is also good to note that these shapes depend on the parameter values of the distributions. It follows that; these compound distributions could be useful in modelling real life phenomena whose failure rate is inverted bathtub, decreasing and increasing. This is unlike the Exponential distribution which has a constant failure rate.

To determine the possible limitations of these newly derived compound distributions, secondary data sets have been collected and the derived compound distributions have been fitted to these data sets as contained in Chapter four. The compound distributions perform better than their sub-models for each of the cases considered (which is in accordance with many other results as presented in Chapter two of this study) except for cases where outliers are present in the data set which results into over-dispersion in the data set. This latter statement is not in agreement with many results in the literature as most authors prefer to show the strength of their newly developed model, leaving aside the possible shortfall(s).

Another generalised family of distributions called the Exponential generalised family of distributions has also been developed and explored. It was noted that it is a sub-model of the Weibull-G family of distributions that was proposed by Bourguignon *et al.*, (2014). Further, it has been used in this research to generalise the Inverse Exponential distribution among other distributions. Its flexibility has been assessed and determined with the aid of applications to real life data sets and a simulation study was conducted.

The aim of the simulation is to know how the parameters of Exponential Inverse Exponential distribution would behave as sample size n increases with respect to biasedness, Standard Error (SE) and Mean Square Error (MSE). Weibull distribution was used to simulate a data set of size $m = 1,000$ and six different samples of sizes $n = 50, 100, 150, 200, 500$ and 800 were drawn using a technique known as simple random sampling (SRS). The Exponential Inverse Exponential was fitted to the simulated data and its parameters were estimated. It was concluded that the standard error generated reduces as the sample size increases. Also, the value for the bias generated was small.

5.3 Recommendations

This study recommends the following:

i More link functions which will result in generalised families of distributions with only one additional shape parameter should be developed.

ii Generalised families of distributions are usually derived from the logit of random variables based on a particular distribution. This study recommends that the cdf of such distribution should be tractable enough and be void of special functions.

5.4 Contributions to Knowledge

This study has led to the following contributions;

i A statistical table for the Inverse Exponential distribution has been developed which would be useful for stakeholders such as students and researchers;

ii Various generalisations that extend the Inverse Exponential distribution have been developed;

iii The study revealed that compound distributions may not be able to model over-dispersed data sets or those with outliers;

iv Simulation of data using another distribution as the basis has been performed; and

v Algorithm was developed in R-programming as codes to achieve the results in the study.

5.5 Further Study

This study can further be extended as follows:

i Other generalised families of distributions (different from the ones used in this study) can be used to generalise the Inverse Exponential distribution.

ii Statistical table for the cdf of the Inverse Exponential distribution could be developed.

iii Other goodness of fit tests could be used to validate the results in this study.

iv Some further properties of the derived compound distributions could be explored.

v Different forms of the Exponential Generalised family of distributions generated from another link function other than the one used in this thesis can be developed. Just the same way there are varieties like Weibull-G and Weibull-X families of distributions in the literature.

vi The simulation study could be performed using the Exponential Inverse Exponential distribution instead of Weibull distribution as used in this study.

vii Simulation studies could be performed for all the newly developed compound distributions in this study.

5.6 Summary

A statistical table for the Inverse Exponential distribution has been generated and this confirms that the shape of the Inverse Exponential distribution is unimodal. The pdf and cdf (including the various statistical properties) of the KIE, TIE, EGIE and WIE distributions have been successfully derived. These compound distributions perform better than the Inverse Exponential distribution (except when the data set is over-dispersed because of outliers in the data set). The log-likelihood values and the AIC values posed by these distributions were used as bases for judgment; the higher the log-likelihood value, the better the distribution and the lower the AIC value, the better the distribution.

The Exponential-G family of distributions have been studied. It was used to extend the Weibull, Pareto, Kumaraswamy and the Inverse Exponential distributions. It was demonstrated that the Exponential Inverse Exponential (EIE) distribution is an improvement over the Inverse Exponential distribution. For the simulation study, it was discovered that the standard error reduces as the sample size increases. In addition, the bias generated is not large.

R software was used to perform all the analyses in this study including the plotting of graphs, simulation study and the newly generated statistical table. Also, some useful R code that were used for the analysis have been provided as Appendix A, B, C, D and E. This will be of help to other researchers and students.

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APPENDIX A: R-CODE THAT GENERATED THE STATISTICAL TABLE FOR THE INVERSE EXPONENTIAL DISTRIBUTION WHEN PARAMETER $\theta = 0.5$

x=seq(1,25,1)

x

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

theta=0.5

```
pdf=(theta/x^2)*exp(-theta/x)
```

probability

0.3032653299 0.0973500979 0.0470267625 0.0275780282 0.0180967484

0.0127783946 0.0095006406 0.0073391646 0.0058392560 0.0047561471

0.0039486076 0.0033305189 0.0028469489 0.0024615203 0.0021493691

0.0018930337 0.0016799594 0.0015009328 0.0013490685 0.0012191374

0.0011071108 0.0010098441 0.0009248539 0.0008501581 0.0007841589

APPENDIX B: R-CODE FOR PLOTTING THE PDF OF THE KIE DISTRIBUTION

```
a=2
b=2
c=3
x=seq(0,15,0.1)
kie.pdf=function(x,a,b,c)
f=a*b*(c/x^2)*((exp(-c/x))^a)*(1-(exp(-c/x)^a))^(b-1)
plot(x,kie.pdf(x,2,2,3),type='l',ylim=c(0,0.8),xlab="x",ylab="pdf")
lines(x,kie.pdf(x,2,2,0.7),col=2)
lines(x,kie.pdf(x,0.6,2,3),col=3)
lines(x,kie.pdf(x,0.2,0.5,0.8),col=4)
legend("topright",inset=0.02,c("a=2.0,b=2.0,c=3.0","a=2.0,b=2.0,c=0.7",
 "a=0.6,b=2.0,c=3.0","a=0.2,b=0.5,c=0.8"),col=1:2:3:4)
```

APPENDIX C: R-CODE FOR ANALYZING DATA III USING KIE, GIE AND IE DISTRIBUTIONS

```
DEATHTIMES<-
c(1,3,3,4,10,13,13,16,16,24,26,27,28,30,30,32,41,51,61,65,67,70,72,73,74,77,79,80,81,
87,87,88,89,91,93,96,97,100,101,104,104,108,109,120,131,150,157,167,231,240,400)
summary(DEATHTIMES)
Min. 1st Qu. Median Mean 3rd Qu. Max.
1.00 29.00 77.00 80.73 100.50 400.00
n=length(DEATHTIMES)
n
51
local(pkg <- select.list(sort(.packages(all.available = TRUE)),graphics=TRUE)
+ if(nchar(pkg)) library(pkg, character.only=TRUE))
Loading required package: miscTools
Warning messages:
1: package maxLik was built under R version 2.14.2
2: package miscTools was built under R version 2.14.2
```

```
loglikkie<-function(p)
n*log(p[1])+n*log(p[2])+n*log(p[3])-2*sum(log(DEATHTIMES))-
(p[1]*p[3]*sum(1/DEATHTIMES))+(p[2]-1)*sum(log(1-exp(-
p[1]*(p[3]/DEATHTIMES))))
d<-maxLik(loglikkie,start=c(0.05,1,1))
There were 50 or more warnings (use warnings() to see the first 50)
summary(d)
```

```
Maximum Likelihood estimation
Newton-Raphson maximisation, 12 iterations
Return code 2: successive function values within tolerance limit
Log-Likelihood: -300.856
3 free parameters
```

Estimates:

Estimate Std. error t value Pr(> t)

0.51378 0.13325 3.8559 0.0001153 ***

0.57711 0.10230 5.6411 1.689e-08 ***

22.03028 2.51599 8.7561 <2.2e-16 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

AIC(d)

607.712

For GIE distribution;

loglikgie<-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(DEATHTIMES))-(p[1]*p[2]*sum(1/DEATHTIMES))

e<-maxLik(loglikgie,start=c(0.05,20))

There were 50 or more warnings (use warnings() to see the first 50)

summary(e)

Maximum Likelihood estimation

Newton-Raphson maximisation, 10 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -306.1066

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

2.4352e-02 3.4117e-03 7.1379 9.474e-13 ***

7.1367e+02 2.9671e+00 240.5242 <2.2e-16 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

AIC(e)

616.2133

For IE distribution;

loglikie<-function(p)

n*log(p[1])-2*sum(log(DEATHTIMES))-(p[1]*sum(1/DEATHTIMES))

f<-maxLik(loglikie,start=c(20))

There were 50 or more warnings (use warnings() to see the first 50)

summary(f)

Maximum Likelihood estimation

Newton-Raphson maximisation, 6 iterations

Return code 1: gradient close to zero

Log-Likelihood: -306.1066

1 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

17.3795 2.4216 7.1769 7.131e-13 *** —

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

AIC(f)

614.2133

APPENDIX D: R-CODE FOR THE ANALYSIS OF EGIE DISTRIBUTION

```
DATAIX<-c(0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451,
0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753,
0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113,
0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570,
1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880,
1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746,
1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316,
1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100,
2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260,
2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455,
3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960)
summary(DATAIX)
Min. 1st Qu. Median Mean 3rd Qu. Max.
0.0251 0.9048 1.7360 1.9590 2.2960 9.0960
local(pkg <- select.list(sort(.packages(all.available =
TRUE)),graphics=TRUE)
+ if(nchar(pkg)) library(pkg, character.only=TRUE))
local(pkg <- select.list(sort(.packages(all.available =
TRUE)),graphics=TRUE)
+ if(nchar(pkg)) library(pkg, character.only=TRUE))
This is package 'modeest' written by P. PONCET. For a complete list
of functions, use 'library(help = "modeest")' or 'help.start()'.

Attaching package: modeest
The following object is masked from package:moments:
skewness
local(pkg <- select.list(sort(.packages(all.available =
TRUE)),graphics=TRUE)
+ if(nchar(pkg)) library(pkg, character.only=TRUE))
Loading required package: miscTools
```

Please cite the 'maxLik' package as:

Henningsen, Arne and Toomet, Ott (2011). maxLik: A package for maximum likelihood estimation in R. Computational Statistics 26(3), 443-458. DOI 10.1007/s00180-010-0217-1.

If you have questions, suggestions, or comments regarding the 'maxLik' package, please use a forum or 'tracker' at maxLik's R-Forge site:

https://r-forge.r-project.org/projects/maxlik/

var(DATAIX)

2.477415

skewness(DATAIX)

1.940616

attr(,"method")

"moment"

kurtosis(DATAIX)

8.160792

n=length(DATAIX)

n

76

---

loglikegie<-function(p)

n*log(p[1])+n*log(p[2])+n*log(p[3])-2*sum(log(DATAIX))-sum(p[3]/DATAIX)+ (p[1]-1)*sum(log(1-exp(-p[3]/DATAIX)))+(p[2]-1)*sum(log(1-(1-exp(-p[3]/DATAIX))^{p}[1]))

d<-maxLik(loglikegie,start=c(1,1,0.05))

There were 50 or more warnings (use warnings() to see the first 50)

summary(d)

Maximum Likelihood estimation

Newton-Raphson maximisation, 123 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -153.583

3 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

7.595e-01 8.057e-02 9.427 ¡2e-16 ***

1.446e+03 4.196e+00 344.622 ¡2e-16 ***

5.649e-05 5.529e-05 1.022 0.307

— Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

AIC(d)

313.166

vcov(d)

[,2] [,3]

6.490878e-03 -5.932001e-03 4.399355e-06

-5.932001e-03 1.760926e+01 -4.959011e-06

4.399355e-06 -4.959011e-06 3.057368e-09

loglikgie<-function(p)

n*log(p[1])+n*log(p[2])-2*sum(log(DATAIX))-sum(p[2]/DATAIX)+
(p[1]-1)*sum(log(1-exp(-p[2]/DATAIX))) g¡-maxLik(loglikgie,start=c(20,1))

There were 50 or more warnings (use warnings() to see the first 50)

summary(g)

Maximum Likelihood estimation

Newton-Raphson maximisation, 8 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -161.9691

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

0.79036 0.12510 6.318 2.65e-10 ***

0.52254 0.09231 5.661 1.51e-08 ***

— Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

AIC(g)

327.9382

vcov(g)

[,2]

0.015650105 0.007957411

0.007957411 0.008520679

loglikie<-function(p) n*log(p[1])-2*sum(log(DATAIX))-sum(p[1]/DATAIX)

h¡-maxLik(loglikie,start=c(0.05))

summary(h)

Maximum Likelihood estimation

Newton-Raphson maximisation, 8 iterations

Return code 1: gradient close to zero

Log-Likelihood: -163.1015

1 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

0.62487 0.07168 8.718 ¡2e-16 ***

— Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

AIC(h)

328.203

APPENDIX E: R-CODE FOR THE SIMULATION

```
population=rweibull(1000,2,3)
x=sample(population,50)
n=length(x)
loglikeie<-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(x))-sum(p[2]/x)-
2*sum(log(1-exp(-p[2]/x)))-(p[1])*sum((exp(-p[2]/x))/(1-exp(-p[2]/x)))
g<-maxLik(loglikeie,start=c(0.05,1))
There were 50 or more warnings (use warnings() to see the first 50)
summary(g)
--------------------------------------------
Maximum Likelihood estimation
Newton-Raphson maximisation, 9 iterations
Return code 1: gradient close to zero
Log-Likelihood: -85.88721
2 free parameters
Estimates:
Estimate Std. error t value Pr(> t)
1.5592 0.6067 2.570 0.0102 *
2.4548 0.6036 4.067 4.76e-05 ***
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
--------------------------------------------
y=sample(population,100)
n=length(y)
loglikeie<-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(y))-sum(p[2]/y)-
2*sum(log(1-exp(-p[2]/y)))-(p[1])*sum((exp(-p[2]/y))/(1-exp(-p[2]/y)))
g<-maxLik(loglikeie,start=c(0.05,1))
There were 50 or more warnings (use warnings() to see the first 50)
summary(g)
--------------------------------------------
Maximum Likelihood estimation
```

Newton-Raphson maximisation, 9 iterations

Return code 1: gradient close to zero

Log-Likelihood: -177.1837

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

1.0018 0.2855 3.509 0.000449 ***

1.7954 0.3573 5.025 5.03e-07 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

---

z=sample(population,150)

n=length(z)

loglikeie<-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(z))-sum(p[2]/z)-2*sum(log(1-exp(-p[2]/z)))-(p[1])*sum((exp(-p[2]/z))/(1-exp(-p[2]/z)))

g<-maxLik(loglikeie,start=c(0.05,1))

There were 50 or more warnings (use warnings() to see the first 50)

summary(g)

Maximum Likelihood estimation

Newton-Raphson maximisation, 14 iterations

Return code 2: successive function values within tolerance limit

Log-Likelihood: -252.4263

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

2.4363 0.5033 4.841 1.29e-06 ***

3.4207 0.3987 8.580 < 2e-16 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

---

k=sample(population,200)

n=length(k)

loglikeie<-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(k))-sum(p[2]/k)-2*sum(log(1-exp(-p[2]/k)))-(p[1])*sum((exp(-p[2]/k))/(1-exp(-p[2]/k)))

g<-maxLik(loglikeie,start=c(0.05,1))

There were 50 or more warnings (use warnings() to see the first 50)

summary(g)

Maximum Likelihood estimation

Newton-Raphson maximisation, 12 iterations

Return code 1: gradient close to zero

Log-Likelihood: -351.3049

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t) 2.0845 0.3905 5.338 9.42e-08 ***

3.2450 0.3572 9.084 < 2e-16 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

---

j=sample(population,500)

n=length(j)

loglikeie¡-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(j))-sum(p[2]/j)-2*sum(log(1-exp(-p[2]/j)))-(p[1])*sum((exp(-p[2]/j))/(1-exp(-p[2]/j)))

g<-maxLik(loglikeie,start=c(0.05,1))

There were 50 or more warnings (use warnings() to see the first 50)

summary(g)

Maximum Likelihood estimation

Newton-Raphson maximisation, 12 iterations

Return code 1: gradient close to zero

Log-Likelihood: -883.4327

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

1.4384 0.1812 7.938 2.05e-15 ***

2.4072 0.1952 12.331 < 2e-16 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

---

i=sample(population,800)

n=length(i)

loglikeie<-function(p) n*log(p[1])+n*log(p[2])-2*sum(log(i))-sum(p[2]/i)-2*sum(log(1-exp(-p[2]/i)))-(p[1])*sum((exp(-p[2]/i))/(1-exp(-p[2]/i)))

g<-maxLik(loglikeie,start=c(0.05,1))

There were 50 or more warnings (use warnings() to see the first 50)

summary(g)

Maximum Likelihood estimation

Newton-Raphson maximisation, 12 iterations

Return code 1: gradient close to zero

Log-Likelihood: -1397.108

2 free parameters

Estimates:

Estimate Std. error t value Pr(> t)

1.6760 0.1616 10.37 <2e-16 ***

2.6778 0.1603 16.70 <2e-16 ***

—

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
