Proceedings of the World Congress on Engineering and Computer Science 2017 Vol I
WCECS 2017, October 25-27, 2017, San Francisco, USA

On Unique Solution of Quantum Stochastic Differential Inclusions

S. A. Bishop, *IAENG Member*, M. C. Agarana, H. I. Okagbue and J. G. Oghonyon

Abstract— **We investigate the existence of solutions of quantum stochastic differential inclusion (QSDI) with some uniqueness properties as a variant of the results in the literature. We impose some weaker conditions on the coefficients and show that under these conditions, a unique solution can be obtained provided the functions $K^P_{\eta\xi}: [0, T] \to \mathbb{R}_+$ are measurable such that their integral is finite.**

Index Terms— **Uniqueness of solution; Weak Lipschitz conditions; stochastic processes. Successive approximations**

I INTRODUCTION

In this paper, we establish existence and uniqueness of solution of the following quantum stochastic differential inclusion (QSDI):

$$x(t) \in a + \int_0^t E(s, x(s))d\Lambda_\pi + F(s, x(s))dA_g(s)$$
$$+G(s, x(s))dA_f^+(s) + H(s, x(s))ds,\ t \in [0, T] \qquad (1)$$

QSDI (1) is understood in the framework of the Hudson and Parthasarathy [9] formulation of Boson quantum stochastic calculus. The maps f, g, π appearing in (1) lie in some suitable function spaces defined in [7]. The integrators $\Lambda_\pi,\ A_f^+ \ and\ A_g$ are the gauge, creation and annihilation processes associated with the basic field operators of quantum field theory defined in [7]. However, in [7] it has been shown that inclusion (1) is equivalent to this first order nonclassical ordinary differential inclusion

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle \in P(t, x)(\eta, \xi)$$
$$x(0) = a,\ \ t \in [0, t] \qquad (2)$$

The map $(\eta, \xi) \to P(t, x)(\eta, \xi)$ appearing in (2) is defined by

$$P(t, y)(\eta, \xi) = (\mu E)(t, y)(\eta, \xi) + (\gamma F)(t, y)(\eta, \xi)$$
$$+ (\sigma G)(t, y)(\eta, \xi) + H(t, y)(\eta, \xi)$$

In [5, 6], some of the results in [2, 7] were generalized. Results on multifunction associated with a set of solutions of non-Lipschitz quantum stochastic differential inclusion (QSDI), which still admits a continuous selection from some subsets of complex numbers were established.

In [3], results on non-uniqueness of solutions of inclusion (2) were established under some strong conditions. Motivated by the results in [5, 6], we establish existence and uniqueness of solution of inclusion (2) under weaker conditions defined in [5]. Here, the map $x \to P(t, x)(\eta, \xi)$ is not necessarily Lipschitz in the sense of [3]. Hence the results here are weaker than the results in [3]. Inclusion (1) has applications in quantum stochastic control theory and the theory of quantum stochastic differential equations with discontinuous coefficients. See [7] and the references therein.

The rest of this paper is organized as follows; Section 3 of this paper will be devoted to the main results of the work while in section 2 some definitions, preliminary results and notations will be presented.

II PRELIMINARY RESULTS

Some of the notations and definitions used here will come from the references [3, 5-7]. $\mathcal{N}$ is a topological space, while clos($\mathcal{N}$), comp($\mathcal{N}$) denote the collection of all nonempty closed, compact subsets of $\mathcal{N}$ respectively. The space $\tilde{\mathcal{A}}$ (a locally convex space) is generated by the family of seminorms $\{\|x\|_{\eta\xi} = \langle \eta, x\xi\rangle,\ x \in \tilde{\mathcal{A}},\ \eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})\}$. $(\tilde{\mathcal{A}}, \tau)$ is the completion of $\mathcal{A}$. Here $\tilde{\mathcal{A}}$ consists of linear operators defined in [3]. In what follows, $\mathbb{D}$ is a pre-Hilbert space, $\mathcal{R}$ its completion, γ a fixed Hilbert space and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable γ- valued maps on $\mathbb{R}_+$. For the definitions and notations of the Hausdorff topology on $\tilde{\mathcal{A}}$ and more see [3] and the references therein.

Definition 1

$\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ is Lipschitzian if

$$\rho_{\eta\xi}(\Phi(t, x) - \Phi(t, y)) \le K^\Phi_{\eta\xi}(t)W(\|x - y\|_{\theta_{\Phi(\eta\xi)}}) \qquad (3)$$

where $W(t) \neq t, I = [0, T] \subseteq \mathbb{R}_+,\ \eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})$.

Remark 2: (i) If $\eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$ and $W(t) = t$ then we obtain the results in [3]

In this case, we obtain a class of multivalued maps which are not necessarily Lipschitzian in the sense of definition (3) (b) in [3].

Definition 3 By a solution of (1) or equivalently (2) we mean a stochastic process $\Phi: I \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying (1).

The following result established in [3] is modified here. However we refer the reader to [3] for a detailed proof as we will only highlight the major changes due to the conditions in this setting.

Theorem 4 Let $B: \mathbb{R}_+ \to L(\tilde{\mathcal{A}})$. For any $x \in \tilde{\mathcal{A}}, L(t, x)$ defined by

$$L(t, x) = \{\ \|B(t)x\|_{\eta\xi}\}\mathcal{N}$$

is Lipschitzian with $W(t) \neq t$.

Proof: We adopt the method of the proof of Theorem 2.2 in [3] as follows:

Let the function $C^B_{\eta\xi}(t) > 0$, then for $x, y \in \tilde{\mathcal{A}},\ t \in \mathbb{R}_+$ we have

$$\rho_{\eta\xi}(L(t, x) - L(t, y)) =$$
$$\rho_{\eta\xi}(\|B(t)x\|_{\eta\xi}\mathcal{N}, \|B(t)y\|_{\eta\xi}\mathcal{N}\ )$$
$$\le \left|\|B(t)x\|_{\eta\xi} - \|B(t)y\|_{\eta\xi}\right|\rho_{\eta\xi}\mathcal{N}$$
$$\le \|\mathcal{N}\|_{\eta\xi}C^B_{\eta\xi}(t)\|x - y\|_{\theta_{B(\eta,\xi)}}$$
$$\le K^L_{\eta\xi}(t)W(\|x - y\|_{\theta_{B(\eta,\xi)}}),$$

Where $K^L_{\eta\xi}(t)W = \|\mathcal{N}\|_{\eta\xi}C^B_{\eta\xi}(t)$.

Manuscript received July 1, 2017; revised July 30, 2017. This work was supported in full by Covenant University.
S. A. Bishop, M. C. Agarana, H. I. Okagbue and J. G. Oghonyon are with the Department of Mathematics, Covenant University, Ota, Nigeria.
sheila.bishop@covenantuniversity.edu.ng
michael.agarana@covenantuniversity.edu.ng
hilary.okagbue@covenantuniversity.edu.ng
godwin.oghonyon@covenantuniversity.edu.ng

Let $\theta_B: (\mathbb{D} \underline{\otimes} \mathbb{E}) \to (\mathbb{D} \underline{\otimes} \mathbb{E})$. So that (2.2) in [3] becomes

$$\rho_{\eta\xi}((\Phi(t,x) - \Phi(t,y)) \leq K^{\Phi}_{\eta\xi}(t)W(\|x-y\|_{\eta,\xi}) \quad (4)$$

$$\eta, \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E}),\ x,\ y \in \tilde{\mathcal{A}},\ W(t) \neq t \text{ and } t \in I.$$

Similarly, (2.5) in [3] becomes

$$\rho_{\eta\xi}(\mathrm{P}(t,x) - \mathrm{P}(t,y)) \leq K^{P}_{\eta\xi}(t)W(\|x-y\|_{\eta,\xi}) \quad (5)$$

For the map $(t,x) \to P(t,x)(\eta,\xi)$. Hence we conclude that the given map is also Lipschitzian.

III MAJOR RESULTS

To establish the major result in this section, we use the methods used in [3, 7]. In the sequel, except otherwise stated, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $t \in [0,1]$ is arbitrary. In line with [5, 6], we make the following assumptions

Let $Z : [0,1] \to \tilde{\mathcal{A}} \in Ad(\tilde{\mathcal{A}})_{wac}$ and for almost all $t \in [0,T]$, there exists $S_{\eta\xi} \in L^1_{loc}([0,T])$ such that

$$\left(\frac{d}{dt}\langle \eta, Z(t)\xi\rangle, P\big(t, Z(t)\big)(\eta,\xi)\right) \leq S_{\eta\xi}(t)$$

Fix $\gamma > 0$ and define the set $Q_{Z,\gamma}$ by,

$$Q_{Z,\gamma} = \{(t,x) \in t \times \tilde{\mathcal{A}} : \|x - Z(t)\|_{\eta\xi} \leq \gamma\}.$$

Since the coefficients E, F, G, H in (1) are Lipschitzian with $Q_{Z,\gamma} \to (clos(\tilde{\mathcal{A}}), \tau_h)$, (5) above holds for a. e.

$$P : \tilde{\mathcal{A}}(\eta,\xi) \to \tilde{\mathcal{A}}(\eta,\xi)$$

I. $\delta_{\eta\xi} \equiv \|x_0 - Z(0)\|_{\eta\xi}$ and $\delta_{\eta\xi} \leq \gamma$

II. $R_{\eta\xi} :=$ max $(\delta_{\eta\xi}, S_{\eta\xi})$ where
$$N_{\eta\xi} = ess\ sup_{[0,1]} S_{\eta\xi}(t)$$

III. Given the sequence $\{(\eta_n, \xi_n) \subseteq (\mathbb{D} \underline{\otimes} \mathbb{E}), n = 1, 2, \ldots\}$, we have $\mathrm{W}\big(sup_{n\in\mathbb{N}}\{ess\ sup_{t\in[0,1]} K^P_{\eta\xi}(t)\}\big) < \infty$

IV. Define $L_{\eta_j\xi_j} := W\left(ess\ sup_{[0,1]} K^P_{\eta_j\xi_j}(t)\right), j \geq 2$

V. Define
$$\varepsilon_{\eta\xi}(t) = 2L_{\eta\xi} + W\left(2L_{\eta\xi}\int_0^t (K^P_{\eta\xi}(s)e^{L_{\eta\xi,s}})ds\right)$$

VI. Define $J \subset [0, 1]$ by
$$\mathrm{J} = \{t \in [0,1]: \varepsilon_{\eta\xi}(t) \leq \gamma\}.$$

We adopt the definitions of following; $L_{\eta\xi}, L_{\eta\xi,n}, R_{\eta_1\xi_1}, L_{\eta,\xi}$ from the reference [3].

Proposition 5. Let $\{\Phi\}_{i=1}^{\infty} \in Ad(\tilde{\mathcal{A}})_{wac}$ be a sequence satisfying;

(i) $(t, \Phi_i(t)) \in Q_{Z,\gamma}, i \geq 1$ for a.e. $t \in$ J.

(ii) There exists $\{V_i\}_{i=1}^{\infty}$ and a constant $L_{\eta\xi} > 0$, such that

(a) $\Phi_i(t) = x_0 + \int_0^t V_{i-1}(s)ds,\ i \geq 1$

(b)
$$\left|\frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_{i-1}(t)\xi\rangle\right| \leq W\left(2L^{i-1}_{\eta\xi}K^P_{\eta\xi}(t)\frac{t^{i-2}}{(i-2)!}\right), \text{ for a. e. } t \in \mathrm{J}.$$

Then,

(c) $\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} \leq W\left(2L_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)\frac{L^{i-2}_{\eta\xi}}{(i-2)!}ds\right),$

where $t \in \mathrm{J}, i \geq 2$.

Proof: The proof is an adaptation of the arguments employed in [3], Proposition 3.1.

Assume (i) and (ii) above hold. Then

$\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} \leq \left|\int_0^t \langle \eta, (\mathrm{V}_{i-1}(s) - \mathrm{V}_{i-2}(s))\xi\rangle ds\right|$ by (ii) in (a)

$$= \left|\int_0^t \frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_{i-1}(t)\xi\rangle ds\right|$$

$$\leq \int_0^t \left|\frac{d}{dt}\langle \eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_{i-1}(t)\xi\rangle\right| ds$$

$$\leq W(2L^{i-1}_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)\frac{s^{i-2}}{(i-2)!}ds)$$

$$= W\left(2L_{\eta\xi}\int_0^t K^P_{\eta\xi}(s)\frac{(L_{\eta\xi}s)^{i-2}}{(i-2)!}ds\right), \qquad t \in J, i \geq 2.$$

The next Theorem is a major result.

Theorem 6. Suppose conditions I- VI hold, $\{E, F, G, H\}: [0,1] \times \tilde{\mathcal{A}} \to (clos(\tilde{\mathcal{A}}), \tau_h)$ is continuous. Then ∃ a unique solution such that

$$\|\Phi(\mathrm{t}) - \mathrm{Z}(\mathrm{t})\|_{\eta\xi} \leq \varepsilon_{\eta\xi}(t), t \in J, \quad (6)$$

$$\left|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Z(t)\xi\rangle\right| \leq L_{\eta\xi}(1 + W(2K^P_{\eta\xi}(t)e^{L_{\eta\xi}t}) \quad (7)$$

Proof. From the references [3, 7, 8], we construct a τ_W-Cauchy sequence $\Phi_n(t)$ of successively approximates $\Phi(t)$. We make the following assumptions:

$\left\{\frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle\right\}_{n\geq 0}$ is Cauchy in $\mathbb{C}$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. $\Phi_0(t) = Z$ is adapted.

By Theorem 1.14.2 in [1], there exists a measurable selection $\mathrm{V}_0(.)(\eta,\xi) \in P(., \Phi_0(.))(\eta,\xi)$ so that (3.3) in [3] holds and since $\mathrm{V}_0(.)(\eta,\xi)$ is locally absolutely integrable, then $\mathrm{V}_0 \in L^1_{loc}(\tilde{\mathcal{A}})$.

Let $\Phi_1(t)$ be defined as

$$\Phi_1(t) = X_0 + \int_0^t V_0(s)ds \quad (8)$$

If $V_0(t) \in \tilde{\mathcal{A}}$, then $\Phi_1(t) \in \tilde{\mathcal{A}}_t$. It implies that (3.4) and (3.5) in [3] hold in this case.

Again ∃ a measurable selection

$$V_1(.)(\eta,\xi) \in P(., V_1(.))(\eta,\xi)$$

which yields

$$\left|V_1(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle\right| = d\left(\frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle, P(t, \Phi_1(\mathrm{t}))(\eta,\xi)\right)$$

$$\leq \rho\left(P\big(t, \Phi_1(t)\big)(\eta,\xi), P\big(t, \Phi_0(t)\big)(\eta,\xi)\right)$$

$$\leq W\left(K^P_{\eta\xi}(t)\|\Phi_1(t) - \Phi_0(t)\|_{\eta_1\xi_1}\right)$$

$$\leq W\left(K^P_{\eta\xi}(t)(\delta_{\eta_1\xi_1} + \int_0^t S_{\eta_1\xi_1}(s)ds)\right) \quad (9)$$

For some $(\eta,\xi), (\eta_1, \xi_1) \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Similarly, for $\mathrm{V}_0(.)$, ∃ $\mathrm{V}_1 \in L^1_{loc}(\tilde{\mathcal{A}})$ resulting in;

$$\mathrm{V}_1(\mathrm{t})(\eta,\xi) = \langle \eta, \mathrm{V}_1(t)\xi\rangle, t \in J \quad (10)$$

Define $\Phi_2(t)$ as $\Phi_1(t)$ in (8) above. Then $\Phi_2(t) \in \tilde{\mathcal{A}}_t$ since $\mathrm{V}_1 \in \tilde{\mathcal{A}}$ and hence $\Phi_2(t)$ is adapted.

Now if we consider t ∈ J, we get,

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} = \left\|\int_0^t (\mathrm{V}_1(s) - \mathrm{V}_0(s))ds\right\|_{\eta\xi}$$

$$= |\int_0^t \langle \eta, (\mathrm{V}_1(s) - \mathrm{V}_0(s))\xi\rangle ds|$$

$$\leq \int_0^t \rho(\mathrm{P}(\mathrm{t}, \Phi_1(s))(\eta,\xi), P\big(t, \Phi_0(s)\big)(\eta,\xi)ds$$

$$\leq W(\int_0^t \mathrm{K}^{\mathrm{P}}_{\eta\xi}(s)\|\Phi_1(s) - \Phi_0(s)\|_{\eta_1\xi_1} ds) \quad (11)$$

By (3.4) in [3], we obtain

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi}\mathrm{W}(\int_0^t \mathrm{K}^{\mathrm{P}}_{\eta\xi}(s)[\delta_{\eta_1\xi_1} + \int_0^t + \mathrm{S}_{\eta_1\xi_1}(\mathrm{r})\,\mathrm{dr}])\mathrm{ds}) \quad (12)$$

Continuing in this manner and replacing 1 with 2 in $\mathrm{V}_1(.)$ for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ we get

$$\left|\mathrm{V}_2(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle\right| = d\left(\frac{d}{dt}\langle \eta, \Phi_2(t)\xi\rangle, P(t, \Phi_2(t))(\eta,\xi)\right)$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\leq \rho(P(t,\Phi_2(t))(\eta,\xi), P(t,\Phi_1(t))$$
$$\leq W(K^{P}_{\eta\xi}(t)\|\Phi_2(t)-\Phi_1(t)\|_{\eta_2\xi_2}$$
$$\leq W^2(K^{P}_{\eta\xi}(t)\int_0^t(K^{P}_{\eta_2\xi_2}(s)[\delta_{\eta_1\xi_1}+\int_0^s S_{\eta_1\xi_1}(r)dr])ds) \quad (13)$$

by (12).

In a similar way we can show that since $V_2(t), V_3(t) \in L^1_{loc}(\tilde{\mathcal{A}})$ there exist $\Phi_3(t), \Phi_4(t)$ defined by

$$\Phi_3(t) = X_0 + \int_0^t V_2(s)ds, \quad t \in J$$

and

$$\Phi_4(t) = X_0 + \int_0^t V_3(s)ds, \quad t \in J \qquad (14)$$

satisfying

$$\|\Phi_3(t)-\Phi_2(t)\|_{\eta\xi} = \left\|\int_0^t (V_2(s)-V_1(s))ds\right\|_{\eta\xi}$$
$$\leq W^2(\int_0^t K^P_{\eta\xi}(s)[\int_0^s K^P_{\eta_2\xi_2}(s')[\delta_{\eta_1\xi_1} + \int_0^{s'} S_{\eta_1\xi_1}(r)dr]ds'])ds)$$
$$= W^2(\int_0^t K^P_{\eta\xi}(s)\int_0^s \delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s')\,ds'ds$$
$$+ W^2(\int_0^t K^P_{\eta\xi}(s)\int_0^s K^P_{\eta_2\xi_2}(s')\int_0^{s'} S_{\eta_1\xi_1}(r)dr\,ds'ds \quad (15)$$

and

$$\|\Phi_4(t)-\Phi_3(t)\|_{\eta\xi} = \left\|\int_0^t (V_3(s)-V_2(s))ds\right\|_{\eta\xi}$$

$$\leq W^3\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s K^P_{\eta_3\xi_3}(s')\int_0^{s'}\delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s'')\,ds''ds'ds\right)$$
$$+ W^3\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s K^P_{\eta_3\xi_3}(s')\int_0^{s'}\delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s'')\right)$$
$$\times\left(\int_0^{s''} S_{\eta_1\xi_1}(r)dr\,ds''ds'ds\right) \qquad (16)$$

Then

$$\left|\frac{d}{dt}\langle\eta,\Phi_4(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_3(t)\xi\rangle\right|$$
$$\leq W^3\left(K^P_{\eta\xi}(t)\int_0^t K^P_{\eta_3\xi_3}(s)\int_0^s \delta_{\eta_1\xi_1}K^P_{\eta_2\xi_2}(s')\,ds'ds\right)$$
$$\leq W^3\left(K^P_{\eta\xi}(t)\int_0^t K^P_{\eta_3\xi_3}(s)\int_0^s K^P_{\eta_2\xi_2}(s')\int_0^{s'} S_{\eta_1\xi_1}\,drds'ds\right) \qquad (17)$$

and by (16) and (17) we get

$$\|\Phi_4(t)-\Phi_3(t)\|_{\eta\xi}$$
$$\leq W\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s L_{\eta_3\xi_3}\int_0^{s'}\delta_{\eta_1\xi_1}L_{\eta_2\xi_2}\,ds''ds'ds\right)$$
$$+W\left(\int_0^t K^P_{\eta\xi}(s)\int_0^s L_{\eta_3\xi_3}\int_0^{s'} L_{\eta_2\xi_2}\int_0^{s''} S_{\eta_1\xi_1}(r)drds''ds'ds\right)$$
$$2L^3_{\eta\xi}W(\int_0^t K^P_{\eta\xi}(s)\frac{s^2}{2}ds)$$

Also for $t \in [0,T]$,

$$\left|\frac{d}{dt}\langle\eta,\Phi_4(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_3(t)\xi\rangle\right|$$
$$\leq W\left(K^P_{\eta\xi}(t)[\delta_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\int_0^t\int_0^s ds'ds]\right)$$
$$+W\left(K^P_{\eta\xi}(t)[S_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\int_0^t\int_0^s\int_0^{s'} drds'ds]\right)$$
$$= W\left(K^P_{\eta\xi}(t)[\delta_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\frac{t^2}{2} + S_{\eta_1\xi_1}L_{\eta_2\xi_2}L_{\eta_3\xi_3}\frac{t^3}{6}]\right)$$
$$\leq W\left(K^P_{\eta\xi}(t)[L^3_{\eta\xi,3}\frac{t^2}{2} + L^3_{\eta\xi,3}\frac{t^3}{6}]\right)$$
$$\leq 2W\left(K^P_{\eta\xi}(t)L^3_{\eta\xi,3}\frac{t^2}{2}\right) \qquad (18)$$

Next, we claim that the sequence $\{\Phi_i(t)\}_{i\geq 1}$ exists. To prove this claim, we let $\Phi_{n+i} : J \to \tilde{\mathcal{A}}$ and by Theorem (1.14.2) in [1], $\exists$ a computable selection $V_n(.)(\eta,\xi) \in P(.,\Phi_n(.))(\eta,\xi)$ which yields

$$\left|V_n(t)(\eta,\xi) - \frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle\right|$$
$$= d\left(\frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle, P(t,\Phi_n(t))(\eta,\xi)\right)$$

Since $(\eta,\xi) \to V_n(t)(\eta,\xi)$, $\exists\ V_n \in L^1_{loc}(\tilde{\mathcal{A}})$ defined by (10) a.e. on J and define $\Phi_{n+1}(t)$ as in (8). Then for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we get,

$$\left|\frac{d}{dt}\langle\eta,\Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_n(t)\xi\rangle\right|$$
$$= \langle\eta,\Phi_n(t)\xi\rangle - \langle\eta,\Phi_{n-1}(t)\xi\rangle$$
$$\leq \rho(P(t,\Phi_n(t))(\eta,\xi), P(t,\Phi_{n-1}(t))(\eta,\xi))$$
$$\leq W(K^P_{\eta\xi}(t)\|\Phi_n(t)-\Phi_{n-1}(t)\|_{\eta\xi})$$
$$\leq 2W\left(K^P_{\eta\xi}(t)[L_{\eta\xi}\int_0^t K^P_{\eta_n\xi_n}(s)\frac{(L_{\eta\xi}s)^{n-2}}{(n-2)!}ds]\right)$$
$$\leq W\left(L^n_{\eta\xi}(t)K^P_{\eta\xi}(t)\frac{(t)^{n-1}}{(n-1)!}\right)$$

This establishes (ii)(b) of Prop. 5. Now, for $t \in J$, we get

$$\|\Phi_{n+1}(t)-\Phi_0(t)\|_{\eta\xi} \leq \|\Phi_1(t)-\Phi_0(t)\|_{\eta\xi}$$
$$+\|\Phi_2(t)-\Phi_1(t)\|_{\eta\xi}$$
$$+\cdots+\|\Phi_{n+1}(t)-\Phi_n(t)\|_{\eta\xi}$$
$$\leq 2L_{\eta\xi}[1+W(\sum_{k=0}^{n-1}\int_0^t K^P_{\eta\xi}(s)\frac{(L_{\eta\xi}s)^k}{k!}ds)]$$
$$\leq 2L_{\eta\xi}(1+W(\int_0^t K^P_{\eta\xi}(s)e^{L_{\eta\xi}s}ds) \leq \gamma \quad (19)$$

So that (6) of theorem 6 and (i) of proposition 5 follows. (ii)(b) of proposition 5 yields

$$\left|\frac{d}{dt}\langle\eta,\Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle\right|$$
$$\leq \left|\frac{d}{dt}\langle\eta,\Phi_1(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle\right|$$
$$+W(\sum_{k=0}^{n-1}2L_{\eta\xi}K^P_{\eta\xi}(t)\frac{(L_{\eta\xi}t)^k}{k!})$$
$$+2L_{\eta\xi}(1+W(K^P_{\eta\xi}(t)e^{L_{\eta\xi}t})$$

Taking the limit as $n\to\infty$, (7) of Theorem 6 follows. Hence $\{\Phi_n(t)\}$ is a Cauchy sequence in $\tilde{\mathcal{A}}$ and converges to $\Phi(t)$. Since $\Phi_n(t) \in Ad(\tilde{\mathcal{A}})_{wac}$, it implies that $\Phi(t) \in Ad(\tilde{\mathcal{A}})_{wac}$.

Remark 7. The result of Corollary 3.4 and Theorem 3.4 in [3] fails in this case since the Lipschitz function is independent of t and $W(t) \neq t$ will not be applicable. Hence we establish our result on uniqueness.

Uniqueness of solution

To establish this result, we assume that $\overline{\Phi}(t) \in Ad(\tilde{\mathcal{A}})_{wac}, t \in [0,T]$ is another solution with $\overline{\Phi}(0) = X_0$. By using equation (8) and hypothesis (iii) in Theorem 6 (see also equation (2.1) in [4]), we obtain

$$\|\Phi(t)-\overline{\Phi}(t)\|_{\eta\xi} = \left\|\int_0^t \left(V(s)-\overline{V}(s)\right)ds\right\|_{\eta\xi}$$
$$= \left|\int_0^t \langle\eta,\left(V(s)-\overline{V}(s)\right)\xi\rangle\,ds\right|$$
$$\leq \int_0^t |\langle\eta,V(s)\xi\rangle - \langle\eta,\overline{V}(s)\xi\rangle|ds$$
$$\leq \int_0^t \rho\left(\left(s,\Phi(s)\right)(\eta,\xi) - \left(s,\overline{\Phi}(s)\right)(\eta,\xi)\right)ds$$
$$\leq W(\int_0^t K^P_{\eta\xi}(s)\left\|\Phi(s)-\overline{\Phi}(s)\right\|_{\eta\xi}ds)$$

Since the integral $\int_0^t K^P_{\eta\xi}(s)\,ds$ exists on [0, T], it is also essentially bounded on the given interval. Hence, there exists a constant $C_{\eta\xi}$ such that $\operatorname{ess\,sup} K^P_{\eta\xi}(s) = C_{\eta\xi}, s \in [0,T]$. Thus

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$\left\|\Phi(t)-\overline{\Phi}(t)\right\|_{\eta\xi} \leq W\left(C_{\eta\xi}, t\int_0^t \left\|\Phi(s)-\overline{\Phi}(s)\right\|_{\eta\xi} ds\right).$$

By the Gronwall's inequality, we conclude that $\Phi(t)=\overline{\Phi}(t), t\in[0,T]$. Hence the solution is unique.

ACKNOWLEDGEMENT

The authors will like to use this medium to appreciation the management of Covenant University for the financial support given towards this research. The constructive suggestions of the reviewers are also greatly appreciated.

CONFLICT OF INTEREST

The authors declare that no conflict of interest with respect to the publication of this paper.

REFERENCES

[1] J. P. Aubin and A. Cellina, “Differential Inclusions,” Springer-Verlag, Berlin, 1984.
[2] E. O. Ayoola, “Continuous Selections of Solutions Sets of Lipschitzian quantum stochastic differential inclusions,” Int. Journal of Theoretical Physics. Vol. 43, Issue 10, pp. 2041-2058, 2004.
[3] E. O. Ayoola, “quantum stochastic differential inclusions satisfying a general Lipschitz condition,” Dynamic System and Appls. Vol. 17, pp. 487-502, 2008
[4] E. O. Ayoola, “On Convergence of One -Step Schemes for Weak Solutions of Quantum Stochastic Differential Equations,” Acta Applicandae Mathematicae, Vol. 67, pp. 19-58, 2001.
[5] S. A. Bishop and T. A. Anake, “Extension of continuous selection sets to non-lipschitzian quantum stochastic differential inclusion,” Stochastic Analysis and Applications, Vol. 31, Issue 6, pp. 1114-1124, 2013.
[6] S. A. Bishop and E. O. Ayoola, “On Topological Properties of Solution Sets of Non Lipschitzian Quantum Stochastic Differential Inclusions,” Journal of Analysis and Mathematical Physics, Springer Basel (2015) (DOI) 10.1007/s13324-015-0109-1
[7] G. O. S. Ekhaguere, “Lipschitzian quantum stochastic differential inclusions,” Int. J. of Theoretical Physics, Vol. 31 Issue 11, pp. 2003-2025, 1992.
[8] G. O. S. Ekhaguere, “Quantum Stochastic Differential Inclutions of Hypermaximal monotone type,” Int. J. Theoret. Phys. Vol 34, Issue 3, pp. 323-353. 1995.
[9] R. L. Hudson and K. R. Parthasarathy, “Quantum Ito's formulae and stochastic evolutions,” Comm. math. phys. Vol. 93, pp. 301-324, 1984.

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)