Classes of Ordinary Differential Equations Obtained for the Probability Functions of Gompertz and Gamma Gompertz Distributions

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the Gompertz and gamma Gompertz distributions. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distributions. Solutions of these ODEs by using numerous available methods are a new ways of understanding the nature of the probability functions that characterize the distributions. The method can be extended to other probability distributions and can serve an alternative to approximation especially the cases of the quantile and inverse survival functions. Finally, the link between distributions extended to their differential equations as seen in the case of the ODE of the hazard function of the gamma Gompertz and exponential distributions.**

***Index Terms*— Quantile function, Gompertz distribution, inverse survival function, calculus, differentiation, probability density function, gamma.**

I. Introduction

Gompertz distribution is one of the most widely applied continuous distributions, being first popularized by [1] after studying the memoirs of [2] and [3]. Feroze and Aslam [4] looked at the point and interval Bayes estimates of the parameters that characterize the distribution while Jaheen [5] and Wang et al. [6] restricted their study to record statistics. Lenart and Missov [7] investigated the goodness of fit test for the distribution. Statistical inference and application of the distribution to censored data can be seen in [8-10]. Other applications include: modeling of survival rate of vehicles [11], modeling software reliability using Gompertz curve [12], modeling and fitting of parameters of the distribution of some observed endangered species [13], demographic ageing [14], modeling unemployment rate [15], evolutionary theory of ageing [16] and growth analysis [17]. Some of the probability models proposed as the modifications of the parent distribution are: beta-Gompertz distribution [18], generalized Gompertz distribution [19], bivariate Gompertz [20], Gompertz-power series distribution [21], bivariate pseudo-Gompertz distribution [22], weighted Gompertz distribution [23], McDonald Gompertz distribution [24] and Gompertz-G family of distributions [25].

Gamma Gompertz distribution was proposed by [26] as a flexible customer lifetime model for modeling customer purchasing behavior. Afshar-Nadjafi [27] developed a computer intensive algorithm for estimating the parameters of the distribution. The distribution is widely applied in the modeling of human mortality [28] and approximations of life expectancies [29].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Gompertz and gamma Gompertz distributions by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [30], beta distribution [31], raised cosine distribution [32], Lomax distribution [33], beta prime distribution or inverted beta distribution [34].

Manuscript received June 30, 2017; revised July 31, 2017. This work was sponsored by Covenant University, Ota, Nigeria.

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II. Gompertz Distribution

Probability Density Function

The probability density function of the Gompertz distribution is given as;

$$f(x)=\beta\eta\,\mathrm{e}^{\beta x}\,\mathrm{e}^{\eta}\,\mathrm{e}^{-\eta\mathrm{e}^{\beta x}} \tag{1}$$

To obtain the first order ordinary differential equation for the probability density function of the Gompertz distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\frac{\beta\,\mathrm{e}^{\beta x}}{\mathrm{e}^{\beta x}}-\frac{\eta\beta\,\mathrm{e}^{\beta x}\,\mathrm{e}^{-\eta\mathrm{e}^{\beta x}}}{\mathrm{e}^{-\eta\mathrm{e}^{\beta x}}}\right\}f(x) \tag{2}$$

The condition necessary for the existence of equation is $\beta,\eta>0, x\geq 0$.

$$f'(x)=(\beta-\eta\beta\,\mathrm{e}^{\beta x})\,f(x) \tag{3}$$

$$f'(x)=\beta(1-\eta\,\mathrm{e}^{\beta x})\,f(x) \tag{4}$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Differentiate further to obtain a simplified ordinary differential equation.

$$f''(x)=\beta(1-\eta\,e^{\beta x})f'(x)-\eta\beta^2 e^{\beta x}f(x) \quad (5)$$

The condition necessary for the existence of equation is $\beta,\eta>0, x\geq 0.$

The following equations obtained from (4) are needed to simplify equation (5);

$$\frac{f'(x)}{f(x)}=\beta(1-\eta\,e^{\beta x}) \quad (6)$$

$$\beta f'(x)=\beta^2 f(x)-\eta\beta^2 e^{\beta x}f(x) \quad (7)$$

$$-\eta\beta^2 e^{\beta x}f(x)=\beta f'(x)-\beta^2 f(x) \quad (8)$$

Substitute equations (6) and (8) into equation (5);

$$f''(x)=\frac{f'^2(x)}{f(x)}+\beta f'(x)-\beta^2 f(x) \quad (9)$$

The second order ordinary differential equation for the probability density function of the Gompertz distribution is given by;

$$f(x)f''(x)-f'^2(x)-\beta f(x)f'(x)+\beta^2 f^2(x)=0 \quad (10)$$

$$f(0)=\beta\eta \quad (11)$$

$$f'(0)=(1-\eta)\eta\beta^2 \quad (12)$$

The parameter β mostly determine the nature of the differential equation.

Case 1, when $\beta=1$, equation (10) becomes;

$$f_a(x)f_a''(x)-f_a'^2(x)-f_a(x)f_a'(x)+f_a^2(x)=0 \quad (13)$$

Case 2, when $\beta=2$, equation (10) becomes;

$$f_b(x)f_b''(x)-f_b'^2(x)-2f_b(x)f_b'(x)+4f_b^2(x)=0 \quad (14)$$

To obtain the third order ordinary differential equation for the probability density function of the Gompertz distribution, differentiate equation (5) to obtain;

$$\begin{aligned}f'''(x)&=\beta(1-\eta\,e^{\beta x})f''(x)\\&-2\eta\beta^2 e^{\beta x}f'(x)-\eta\beta^3 e^{\beta x}f(x)\end{aligned} \quad (15)$$

The following equations obtained from (4) are needed to simplify equation (15);

Modify equation (8), to become;

$$-2\eta\beta^2 e^{\beta x}=\frac{\beta f'(x)-\beta^2 f(x)}{f(x)} \quad (16)$$

$$-\eta\beta^3 e^{\beta x}f(x)=\beta^2 f'(x)-\beta^3 f(x) \quad (17)$$

Substitute equations (6), (16) and (17) into equation (15);

$$\begin{aligned}f'''(x)&=\frac{f'(x)f''(x)}{f(x)}+\frac{(\beta f'(x)-\beta^2 f(x))f'(x)}{f(x)}\\&+(\beta^2 f'(x)-\beta^3 f(x))\end{aligned} \quad (18)$$

$$\begin{aligned}&f(x)f'''(x)-f'(x)f''(x)-(\beta f'(x)\\&-\beta^2 f(x))f'(x)-(\beta^2 f'(x)-\beta^3 f(x))f(x)=0\end{aligned} \quad (19)$$

$$f''(0)=\frac{f'^2(0)+\beta f(0)f'(0)-\beta^2 f^2(0)}{f(0)} \quad (20)$$

$$f''(0)=\frac{(1-\eta)^2\eta^2\beta^4+(1-\eta)\eta^2\beta^4-\eta^2\beta^4}{\eta\beta} \quad (21)$$

$$\begin{aligned}f''(0)&=(1-\eta)^2\eta\beta^3+(1-\eta)\eta\beta^3-\eta\beta^3\\&=\eta(\eta^2-3\eta+1)\beta^3\end{aligned} \quad (22)$$

QUANTILE FUNCTION

The Quantile function of the Gompertz distribution is given as ;

$$Q(p)=\frac{1}{\beta}\ln\left(1-\frac{1}{\eta}\ln(1-p)\right) \quad (23)$$

Differentiate equation (23);

$$Q'(p)=\frac{1}{\beta\eta(1-p)\left(1-\frac{1}{\eta}\ln(1-p)\right)} \quad (24)$$

The condition necessary for the existence of equation is $\beta,\eta>0, 0<p<1.$

Differentiate further to obtain a simplified ordinary differential equation.

$$Q''(p)=\left\{-\frac{(1-p)^{-2}}{(1-p)^{-1}}-\frac{\frac{1}{\eta(1-p)}\left(1-\frac{1}{\eta}\ln(1-p)\right)^{-2}}{\left(1-\frac{1}{\eta}\ln(1-p)\right)^{-1}}\right\}Q'(p) \quad (25)$$

$$Q''(p)=-\left\{\frac{1}{1-p}+\frac{1}{\eta(1-p)\left(1-\frac{1}{\eta}\ln(1-p)\right)}\right\}Q'(p) \quad (26)$$

The condition necessary for the existence of equation is $\beta,\eta>0, 0<p<1.$

Equation (24) can be modified to become;

$$\beta Q'(p)=\frac{1}{\eta(1-p)\left(1-\frac{1}{\eta}\ln(1-p)\right)} \quad (27)$$

Substitute equation (27) into equation (26);

$$Q''(p)=-\left[\frac{1}{1-p}+\beta Q'(p)\right]Q'(p) \quad (28)$$

The second order ordinary differential equation for the Quantile function of the Gompertz distribution is given by;

$$(1-p)Q''(p)+Q'(p)+(1-p)\beta Q'^2(p)=0 \quad (29)$$

$$Q(0.1)=\frac{1}{\beta}\ln\left(1+\frac{0.1054}{\eta}\right) \quad (30)$$

$$Q'(0.1)=\frac{10}{9\beta(\eta+0.1054)} \quad (31)$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

SURVIVAL FUNCTION

The survival function of the Gompertz distribution is given as ;

$$S(t)=e^{-\eta(e^{\beta t}-1)} \tag{32}$$

Differentiate equation (32);

$$S'(t)=-\eta\beta e^{\beta t} e^{-\eta(e^{\beta t}-1)} \tag{33}$$

The condition necessary for the existence of equation is $\beta,\eta>0, t\geq 0$.

$$S'(t)=-f(t) \tag{34}$$

Substitute equation (32) into (33);

$$S'(t)=-\beta\eta e^{\beta t} S(t) \tag{35}$$

Differentiate equation (35);

$$S''(t)=-\beta\eta(e^{\beta t} S'(t)+\beta e^{\beta t} S(t)) \tag{36}$$

The condition necessary for the existence of equation is $\beta,\eta>0, t\geq 0$.

$$S''(t)=-\beta\eta e^{\beta t}(S'(t)+\beta S(t)) \tag{37}$$

Equation (35) can also be written as;

$$\frac{S'(t)}{S(t)}=-\beta\eta e^{\beta t} \tag{38}$$

Substitute equation (38) into (37);

$$S''(t)=\frac{S'(t)}{S(t)}(S'(t)+\beta S(t)) \tag{39}$$

The second order ordinary differential equation for the survival function of the Gompertz distribution is given by;

$$S(t)S''(t)-S'^2(t)-\beta S(t)S'(t)=0 \tag{40}$$

$$S(0)=1 \tag{41}$$

$$S'(0)=-\eta\beta \tag{42}$$

Alternatively the ordinary differential equation can be obtained from the further evaluation of equation (34).

INVERSE SURVIVAL FUNCTION

The inverse survival function of the Gompertz distribution is given as ;

$$Q(p)=\frac{1}{\beta}\ln\left(1-\frac{1}{\eta}\ln p\right) \tag{43}$$

Differentiate equation (43);

$$Q'(p)=-\frac{1}{\beta\eta p\left(1-\frac{1}{\eta}\ln p\right)} \tag{44}$$

The condition necessary for the existence of equation is $\beta,\eta>0, 0<p<1$.

Differentiate further to obtain a simplified ordinary differential equation.

$$Q''(p)=-\left\{-\frac{p^{-2}}{p^{-1}}+\frac{\frac{1}{\eta p}\left(1-\frac{1}{\eta}\ln p\right)^{-2}}{\left(1-\frac{1}{\eta}\ln p\right)^{-1}}\right\}Q'(p) \tag{45}$$

$$Q''(p)=\left\{\frac{1}{p}-\frac{1}{\eta p\left(1-\frac{1}{\eta}\ln p\right)}\right\}Q'(p) \tag{46}$$

The condition necessary for the existence of equation is $\beta,\eta>0, 0<p<1$.

Equation (44) can be modified to become;

$$\beta Q'(p)=-\frac{1}{\eta p\left(1-\frac{1}{\eta}\ln p\right)} \tag{47}$$

Substitute equation (47) into equation (46);

$$Q''(p)=\left[\frac{1}{p}+\beta Q'(p)\right]Q'(p) \tag{48}$$

The second order ordinary differential equation for the inverse survival function of the Gompertz distribution is given by;

$$(1-p)Q''(p)-Q'(p)-(1-p)\beta Q'^2(p)=0 \tag{49}$$

$$Q(0.1)=\frac{1}{\beta}\ln\left(1+\frac{2.3026}{\eta}\right) \tag{50}$$

$$Q'(0.1)=-\frac{10}{\beta\left(\eta^2+2.3026\right)} \tag{51}$$

HAZARD FUNCTION

The hazard function of the Gompertz distribution is given as;

$$h(t)=\beta\eta e^{\beta t} \tag{52}$$

Differentiate equation (52);

$$h'(t)=\beta^2\eta e^{\beta t} \tag{53}$$

The condition necessary for the existence of equation is $\beta,\eta>0, t\geq 0$.

$$h'(t)=\beta h(t) \tag{54}$$

The first order ordinary differential equation for the hazard function of the Gompertz distribution is given by;

$$h'(t)-\beta h(t)=0$$

(55) $h(0)=\beta\eta$

(56) To obtain the second order ordinary differential equations for the Hazard function of the Gompertz distribution, differentiate equation (53);

$$h''(t)=\beta^3\eta e^{\beta t} \tag{57}$$

$$h'(0)=\beta^2\eta \tag{58}$$

Two ordinary differential equations can be obtained from the simplification of equation (57);

ODE 1: Using equation (52) in equation (57);

$$h''(t)=\beta^2(\beta\eta e^{\beta t})=\beta^2 h(t) \tag{59}$$

$$h''(t)-\beta^2 h(t)=0 \tag{60}$$

ODE 2: Using equation (53) in equation (57);

$$h''(t)=\beta(\beta^2\eta e^{\beta t})=\beta h'(t) \tag{61}$$

$$h''(t)-\beta h'(t)=0 \tag{62}$$

To obtain the third order ordinary differential equations for

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

the Hazard function of the Gompertz distribution, differentiate equation (57);

$$h'''(t)=\beta^4\eta\,\mathrm{e}^{\beta t} \tag{63}$$

$$h''(0)=\beta^3\eta \tag{64}$$

Three ordinary differential equations can be obtained from the simplification of equation (63);

ODE 1: Using equation (52) in equation (63);

$$h'''(t)=\beta^3(\beta\eta\,\mathrm{e}^{\beta t})=\beta^3 h(t) \tag{65}$$

$$h'''(t)-\beta^3 h(t)=0 \tag{66}$$

ODE 2: Using equation (53) in equation (63);

$$h'''(t)=\beta^2(\beta^2\eta\,\mathrm{e}^{\beta t})=\beta^2 h'(t) \tag{67}$$

$$h'''(t)-\beta^2 h'(t)=0 \tag{68}$$

ODE 3: Using equation (57) in equation (63);

$$h'''(t)=\beta(\beta^3\eta\,\mathrm{e}^{\beta t})=\beta h''(t) \tag{69}$$

$$h'''(t)-\beta h''(t)=0 \tag{70}$$

REVERSED HAZARD FUNCTION

The reversed hazard function of the Gompertz distribution is given as;

$$j(t)=\frac{\beta\eta\,\mathrm{e}^{\beta t}\,\mathrm{e}^{\eta}\,\mathrm{e}^{-\eta\,\mathrm{e}^{\beta t}}}{\mathrm{e}^{-(\eta\,\mathrm{e}^{\beta t}-1)}} \tag{71}$$

To obtain the first order ordinary differential equation for the reversed hazard function of the Gompertz distribution, differentiate equation (71), to obtain;

$$j'(t)=\left\{\begin{array}{l}\dfrac{\beta\,\mathrm{e}^{\beta t}}{\mathrm{e}^{\beta t}}-\dfrac{\eta\beta\,\mathrm{e}^{\beta t}\,\mathrm{e}^{-\eta\,\mathrm{e}^{\beta t}}}{\mathrm{e}^{-\eta\,\mathrm{e}^{\beta t}}}\\ +\dfrac{\beta\eta\,\mathrm{e}^{\beta t}\,\mathrm{e}^{\eta}\,\mathrm{e}^{-\eta\,\mathrm{e}^{\beta t}}(\mathrm{e}^{-(\eta\,\mathrm{e}^{\beta t}-1)})^{-2}}{(\mathrm{e}^{-(\eta\,\mathrm{e}^{\beta t}-1)})^{-1}}\end{array}\right\}j(t) \tag{72}$$

The condition necessary for the existence of equation is $\beta,\eta>0, t\geq 0$.

$$j'(t)=\left(\beta-\eta\beta\,\mathrm{e}^{\beta t}+j(t)\right)j(t) \tag{73}$$

Differentiate further to obtain a simplified ordinary differential equation.

$$\begin{aligned}j''(t)&=\left(\beta-\eta\beta\,\mathrm{e}^{\beta t}+j(t)\right)j'(t)\\&\quad+(j'(t)-\eta\beta^2\,\mathrm{e}^{\beta t})\,j(t)\end{aligned} \tag{74}$$

The condition necessary for the existence of equation is $\beta,\eta>0, t\geq 0$.

The following equations obtained from (73) are needed to simplify equation (74);

$$\frac{j'(t)}{j(t)}=\left(\beta-\eta\beta\,\mathrm{e}^{\beta t}+j(t)\right) \tag{75}$$

$$\beta j'(t)=\left(\beta^2-\eta\beta^2\,\mathrm{e}^{\beta t}+\beta j(t)\right)j(t) \tag{76}$$

$$-\eta\beta^2\,\mathrm{e}^{\beta t}\;j(t)=\beta j'(t)-\beta j^2(t)-\beta^2 j(t) \tag{77}$$

Substitute equations (75) and (77) into equation (74);

$$j''(t)=\frac{j'^2(t)}{j(t)}+j(t)j'(t)-\beta j'(t)-\beta j^2(t)-\beta^2 j(t) \tag{78}$$

The second order ordinary differential equation for the probability density function of the Gompertz distribution is given by;

$$\begin{aligned}&j(t)j''(t)-j'^2(t)-j^2(t)j'(t)\\&+\beta j(t)j'(t)+\beta j^3(t)+\beta^2 j^2(t)=0\end{aligned} \tag{79}$$

$$j(0)=\beta\eta \tag{80}$$

$$\begin{aligned}j'(0)&=\left(\beta-\eta\beta+j(0)\right)j(0)\\&=\left(\beta-\eta\beta+\beta\eta\right)\beta\eta=\eta\beta^2\end{aligned} \tag{81}$$

III. GAMMA GOMPERTZ DISTRIBUTION PROBABILITY DENSITY FUNCTION

The probability density function of the Gamma Gompertz distribution is given as;

$$f(x)=\frac{bs\,\mathrm{e}^{bx}\,\beta^s}{(\beta-1+\mathrm{e}^{bx})^{s+1}} \tag{82}$$

When β = 1, equation (82) reduces to the exponential distribution.

To obtain the first order ordinary differential equation for the probability density function of the Gamma Gompertz distribution, differentiate equation (82), to obtain;

$$f'(x)=\left\{\frac{b\,\mathrm{e}^{bx}}{\mathrm{e}^{bx}}+\frac{b(s+1)\mathrm{e}^{bx}(\beta-1+\mathrm{e}^{bx})^s}{(\beta-1+\mathrm{e}^{bx})^{s+1}}\right\}f(x) \tag{83}$$

$$f'(x)=\left\{b+\frac{b(s+1)\mathrm{e}^{bx}}{(\beta-1+\mathrm{e}^{bx})}\right\}f(x) \tag{84}$$

When $\beta=1$, equation (84) becomes;

$$f_a'(x)=[b+b(s+1)]\,f_a(x) \tag{85}$$

$$f_a'(x)-b(s+2)\,f_a(x)=0 \tag{86}$$

To obtain a simplified ordinary differential equations, differentiate equation (84);

$$\begin{aligned}f''(x)&=\left\{b+\frac{b(s+1)\mathrm{e}^{bx}}{(\beta-1+\mathrm{e}^{bx})}\right\}f'(x)\\&+\left\{-\frac{b^2(s+1)(\mathrm{e}^{bx})^2}{(\beta-1+\mathrm{e}^{bx})^2}+\frac{b^2(s+1)\mathrm{e}^{bx}}{(\beta-1+\mathrm{e}^{bx})}\right\}f(x)\end{aligned} \tag{87}$$

The following equations obtained from (63) are needed to simplify equation (87);

$$\frac{f'(x)}{f(x)}=b+\frac{b(s+1)\mathrm{e}^{bx}}{(\beta-1+\mathrm{e}^{bx})} \tag{88}$$

$$b\left(\frac{f'(x)}{f(x)}-b\right)=\frac{b^2(s+1)\mathrm{e}^{bx}}{(\beta-1+\mathrm{e}^{bx})} \tag{89}$$

$$\left(\frac{f'(x)}{f(x)}-b\right)^2=\frac{b^2(s+1)^2(\mathrm{e}^{bx})^2}{(\beta-1+\mathrm{e}^{bx})^2} \tag{90}$$

$$\frac{1}{(s+1)}\left(\frac{f'(x)}{f(x)}-b\right)^2=\frac{b^2(s+1)(\mathrm{e}^{bx})^2}{(\beta-1+\mathrm{e}^{bx})^2} \tag{91}$$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Substitute equations (88), (89) and (91) into equation (87);

$$f''(x)=\frac{f'^2(x)}{f(x)}+\left\{\begin{array}{l}-\frac{1}{(s+1)}\left(\frac{f'(x)}{f(x)}-b\right)^2\\ +b\left(\frac{f'(x)}{f(x)}-b\right)\end{array}\right\}f(x) \quad (92)$$

Simplify to obtain;

$$f''(x)=\frac{(s+1)f'^2(x)-f'^2(x)}{(s+1)f(x)}+\frac{2bf'(x)+b(s+1)f'(x)}{s+1}+\frac{b^2f(x)-b^2(s+1)f(x)}{s+1} \quad (93)$$

$$f''(x)=\frac{sf'^2(x)}{(s+1)f(x)}+\frac{b(s+3)f'(x)}{s+1}-\frac{b^2sf(x)}{s+1} \quad (94)$$

The second order ordinary differential equation for the probability density function of Gamma Gompertz distribution is given by;

$$\begin{array}{l}(s+1)f(x)f''(x)-sf'^2(x)\\ -b(s+3)f(x)f'(x)+b^2sf^2(x)=0\end{array} \quad (95)$$

$$f(1)=\frac{bs\,\mathrm{e}^b\beta^s}{(\beta-1+\mathrm{e}^b)^{s+1}} \quad (96)$$

$$f'(1)=\left(b+\frac{b(s+1)\mathrm{e}^b}{(\beta-1+\mathrm{e}^b)}\right)\frac{bs\,\mathrm{e}^b\beta^s}{(\beta-1+\mathrm{e}^b)^{s+1}} \quad (97)$$

QUANTILE FUNCTION

The Quantile function of the Gamma Gompertz distribution is given as ;

$$Q(p)=\frac{1}{b}\ln\left(1-\beta+\left(\frac{\beta^s}{(1-p)}\right)\right)^{\frac{1}{s}} \quad (98)$$

Differentiate equation (98);

$$Q'(p)=-\frac{\beta^s\left(1-\beta+\left(\frac{\beta^s}{(1-p)}\right)\right)^{\frac{1}{s}-1}}{bs(1-p)^2\left(1-\beta+\left(\frac{\beta^s}{(1-p)}\right)\right)^{\frac{1}{s}}} \quad (99)$$

$$Q'(p)=-\frac{\beta^s}{bs(1-p)^2\left(1-\beta+\left(\frac{\beta^s}{(1-p)}\right)\right)} \quad (100)$$

$$Q'(p)=-\frac{\beta^s}{bs(1-p)\big((1-\beta)(1-p)+\beta^s\big)} \quad (101)$$

The first order ordinary differential equation for the Quantile function of Gamma Gompertz distribution is given by; $bs(1-p)\big((1-\beta)(1-p)+\beta^s\big)Q'(p)+\beta^s=0$

(102) $Q\left(\frac{1}{10}\right)=\frac{1}{b}\ln\left(1-\beta+\left(\frac{10\beta^s}{9}\right)\right)^{\frac{1}{s}}$

(103) When $\beta=1,$ equation (102) becomes;

$$bs(1-p)Q'(p)+1=0$$

(104)

SURVIVAL FUNCTION

The survival function of the Gamma Gompertz distribution is given as;

$$S(t)=\frac{\beta^s}{(\beta-1+\mathrm{e}^{bt})^s} \quad (105)$$

Differentiate equation (105);

$$S'(t)=-\frac{bs\beta^s\,\mathrm{e}^{bt}}{(\beta-1+\mathrm{e}^{bt})^{s+1}} \quad (106)$$

The following equations obtained from (105) are needed to simplify equation (106);

$$(\beta-1+\mathrm{e}^{bt})^s=\frac{\beta^s}{S(t)} \quad (107)$$

$$\beta-1+\mathrm{e}^{bt}=\frac{\beta}{S^s(t)} \quad (108)$$

$$\mathrm{e}^{bt}=1-\beta+\frac{\beta}{S^s(t)} \quad (109)$$

Substitute equations (107), (108) and (19) into equation (106);

$$S'(t)=-\frac{bs\beta^s\left(1-\beta+\frac{\beta}{S^s(t)}\right)}{\left(\frac{\beta^s}{S(t)}\right)\left(\frac{\beta}{S^s(t)}\right)} \quad (110)$$

$$S'(t)=-\frac{bs\big((1-\beta)S^s(t)+\beta\big)S(t)}{\beta} \quad (111)$$

The first order ordinary differential equation for the Survival function of Gamma Gompertz distribution is given by; $\beta S'(t)+bs\big((1-\beta)S^s(t)+\beta\big)S(t)=0$

(112) $S(1)=\frac{\beta^s}{(\beta-1+\mathrm{e}^b)^s}$

(113) When $\beta=1,$ equation (112) becomes;

$$S_a'(t)+bsS_a(t)=0$$

(114)

INVERSE SURVIVAL FUNCTION

The inverse survival function of the Gamma Gompertz distribution is given as ;

$$Q(p)=\frac{1}{b}\ln\left(1-\beta+\left(\frac{\beta^s}{p}\right)\right)^{\frac{1}{s}} \quad (115)$$

Differentiate equation (115);

$$Q'(p)=\frac{\beta^s\left(1-\beta+\left(\frac{\beta^s}{p}\right)\right)^{\frac{1}{s}-1}}{bsp^2\left(1-\beta+\left(\frac{\beta^s}{p}\right)\right)^{\frac{1}{s}}} \quad (116)$$

$$Q'(p)=\frac{\beta^s}{bsp^2\left(1-\beta+\left(\frac{\beta^s}{p}\right)\right)} \quad (117)$$

$$Q'(p)=\frac{\beta^s}{bsp\left((1-\beta)p+\beta^s\right)} \quad (118)$$

The first order ordinary differential equation for the Quantile function of gamma Gompertz distribution is given by; $bsp\left((1-\beta)p+\beta^s\right)Q'(p)-\beta^s=0$

(119) $Q\left(\frac{1}{10}\right)=\frac{1}{b}\ln\left(1-\beta+10\beta^s\right)^{\frac{1}{s}}$

(120) When $\beta=1,$ equation (119) becomes;

$$bspQ'(p)-1=0$$

(121)

HAZARD FUNCTION

The hazard function of the gamma Gompertz distribution is given as;

$$h(t)=\frac{bs\,e^{bt}}{\beta-1+e^{bt}} \quad (122)$$

Differentiate equation (122);

$$h'(t)=\left\{\frac{b\,e^{bt}}{e^{bt}}-\frac{b\,e^{bt}(\beta-1+e^{bt})^{-2}}{(\beta-1+e^{bt})^{-1}}\right\}h(t) \quad (123)$$

$$h'(t)=\left\{b-\frac{b\,e^{bt}}{\beta-1+e^{bt}}\right\}h(t) \quad (124)$$

$$h'(t)=\left\{b-\frac{h(t)}{s}\right\}h(t) \quad (125)$$

The first order ordinary differential equation for the hazard function of gamma Gompertz distribution is given by;

$$sh'(t)+h^2(t)-bsh(t)=0 \quad (126)$$

$$h(1)=\frac{bs\,e^{b}}{\beta-1+e^{b}} \quad (127)$$

When $\beta=1,$ equation (126) becomes;

$$h'(t)=\left\{b-\frac{b\,e^{bt}}{e^{bt}}\right\}h(t)=0 \quad (128)$$

This is because the hazard function of the exponential distribution is constant because of the lack of memory nature of the distribution or constant hazard rate.

REVERSED HAZARD FUNCTION

The reversed hazard function of the gamma Gompertz distribution is given as;

$$j(t)=\frac{bs\,e^{bt}\,\beta^s}{((\beta-1+e^{bt})^s-\beta^s)(\beta-1+e^{bt})} \quad (129)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the gamma Gompertz distribution, differentiate equation (129), to obtain;

$$j'(t)=\left\{\begin{matrix}\frac{b\,e^{bt}}{e^{bt}}-\frac{b\,e^{bt}(\beta-1+e^{bt})^{-2}}{(\beta-1+e^{bt})^{-1}}\\ -\frac{bs\,e^{bt}(\beta-1+e^{bt})^{s-1}((\beta-1+e^{bt})^s-\beta^s)^{-2}}{((\beta-1+e^{bt})^s-\beta^s)^{-1}}\end{matrix}\right\}j(t) \quad (130)$$

$$j'(t)=\left\{b-\frac{bs\,e^{bt}(\beta-1+e^{bt})^{s-1}}{((\beta-1+e^{bt})^s-\beta^s)}-\frac{b\,e^{bt}}{(\beta-1+e^{bt})}\right\}j(t) \quad (131)$$

$$j'(t)=\left\{b-\frac{b\,e^{bt}}{(\beta-1+e^{bt})}-\frac{(\beta-1+e^{bt})^s\,j(t)}{\beta^s}\right\}j(t) \quad (132)$$

The ordinary differential equations can be obtained for the particular values of the parameters or higher orders.

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [35-45]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

IV. CONCLUDING REMARKS

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the Gompertz and gamma Gompertz distributions. The work was simplified by the application of simple algebraic procedures. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs. Furthermore the ODE of the hazard function of the gamma Gompertz distribution for a particular case is related to the exponential just as their respective PDFs.

ACKNOWLEDGEMENT

The authors are unanimous in appreciation of financial sponsorship from Covenant University. The constructive suggestions of the reviewers are greatly appreciated.

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ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)