Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Frĕchet Distribution

Hilary I. Okagbue, Abiodun A. Opanuga, *Member, IAENG,* Enahoro A. Owoloko and Muminu O. Adamu

***Abstract*— In this work, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the exponentiated Frĕchet distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are a new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions and can serve an alternative to approximation.**

***Index Terms*— Exponentiated, Fréchet distribution, hazard function, calculus, differentiation.**

I. Introduction

NADARAJAH and Kotz [1] proposed the distribution as an improved model over the parent Fréchet distribution. The distribution is a sub model of exponentiated Gumbel type-2 Distribution proposed by [2]. The distribution has been applied as a regression model in modeling positive responses [3].

Other exponentiated class of distributions include: exponentiated Weibull [4-6], exponentiated exponential [7], exponentiated generalized inverted exponential distribution [8], exponentiated generalized inverse Gaussian distribution [9], exponentiated inverted Weibull distribution [10-11], gamma-exponentiated exponential distribution [12], exponentiated gamma distribution [13], exponentiated Gumbel distribution [14], exponentiated uniform distribution [15], beta exponentiated Weibull distribution [16], exponentiated log-logistic distribution [17], exponentiated Kumaraswamy distribution [18], exponentiated modified Weibull extension distribution [19] and exponentiated Pareto distribution [20].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of exponentiated Frĕchet distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [21], beta distribution [22], raised cosine distribution [23], Lomax distribution [24], beta prime distribution or inverted beta distribution [25].

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H. I. Okagbue, A. A. Opanuga and E. A. Owoloko are with the Department of Mathematics, Covenant University, Ota, Nigeria.
hilary.okagbue@covenantuniversity.edu.ng
abiodun.opanuga@covenantuniversity.edu.ng
alfred.owoloko@covenantuniversity.edu.ng
M. O. Adamu is with the Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria.

II. Probability Density Function

The probability density function of the exponentiated Frêchet distribution is given as;

$$f(x)=\alpha\lambda\sigma^{\lambda}x^{-(\lambda+1)}\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{\alpha-1} \qquad (1)$$

To obtain the first order ordinary differential equation for the probability density function of the exponentiated Frêchet distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\begin{array}{l}-\dfrac{(\lambda+1)x^{-(\lambda+2)}}{x^{-(\lambda+1)}}+\dfrac{\frac{\lambda}{x}\left(\frac{\sigma}{x}\right)^{\lambda}\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}\\ -\dfrac{\frac{\lambda}{x}\left(\frac{\sigma}{x}\right)^{\lambda}(\alpha-1)\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{\alpha-2}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{\alpha-1}}\end{array}\right\}f(x) \qquad (2)$$

The condition necessary for the existence of equation is $\alpha,\lambda,\sigma,x>0$.

$$f'(x)=\left\{-\frac{(\lambda+1)}{x}+\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{\lambda\sigma^{\lambda}(\alpha-1)\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{x^{\lambda+1}(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}\right\}f(x) \qquad (3)$$

Another process of differentiation is carried out on equation (3) to obtain;

$$f''(x)=\left\{-\frac{(\lambda+1)}{x}+\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{\lambda\sigma^{\lambda}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{x^{\lambda+1}(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}\right\}f'(x)$$
$$+\left(\frac{(\lambda+1)}{x^{2}}-\frac{\lambda(\lambda+1)\sigma^{\lambda}}{x^{\lambda+2}}\right)f(x)$$
$$+\left\{\begin{matrix}-\dfrac{(\alpha-1)(\lambda\sigma^{\lambda}x^{-(\lambda+1)}\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{2}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{2}}\\ +\dfrac{\lambda^{2}(\alpha-1)\sigma^{2\lambda}(x^{-(\lambda+1)})^{2}\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}\end{matrix}\right\}f(x)$$
$$+\left\{\frac{\lambda(\lambda+1)(\alpha-1)\sigma^{\lambda}x^{-(\lambda+2)}\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}\right\}f(x) \tag{4}$$

The condition necessary for the existence of equation is $\alpha,\lambda,\sigma,x>0$.

The following equations obtained from equation (3) are needed to simplify equation (4);

$$\frac{f'(x)}{f(x)}=-\frac{(\lambda+1)}{x}+\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{\lambda\sigma^{\lambda}x^{-(\lambda+1)}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})} \tag{5}$$

$$\frac{\lambda\sigma^{\lambda}x^{-(\lambda+1)}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})}=\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)} \tag{6}$$

$$\frac{(\lambda\sigma^{\lambda}x^{-(\lambda+1)}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{2}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{2}} \tag{7}$$
$$=\left(\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)}\right)^{2}$$

$$\frac{(\alpha-1)(\lambda\sigma^{\lambda}x^{-(\lambda+1)}\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{2}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})^{2}} \tag{8}$$
$$=\frac{1}{\alpha-1}\left(\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)}\right)^{2}$$

$$\frac{\lambda^{2}(\sigma^{\lambda})^{2}x^{-(\lambda+1)}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})} \tag{9}$$
$$=\lambda\sigma^{\lambda}\left(\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)}\right)$$

$$\frac{\lambda^{2}(\sigma^{\lambda})^{2}(x^{-(\lambda+1)})^{2}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})} \tag{10}$$
$$=\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}\left(\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)}\right)$$

$$\frac{\lambda(\lambda+1)\sigma^{\lambda}x^{-(\lambda+1)}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})} \tag{11}$$
$$=(\lambda+1)\left(\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)}\right)$$

$$\frac{\lambda(\lambda+1)\sigma^{\lambda}x^{-(\lambda+2)}(\alpha-1)\,\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}}}{(1-\mathrm{e}^{-\left(\frac{\sigma}{x}\right)^{\lambda}})} \tag{12}$$
$$=\frac{\lambda+1}{x}\left(\frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\frac{(\lambda+1)}{x}-\frac{f'(x)}{f(x)}\right)$$

Substitute equations (5), (8), (10) and (12) into equation (4);

$$f''(x)=\frac{f'^{2}(x)}{f(x)}+\left(\begin{matrix}\dfrac{(\lambda+1)}{x^{2}}-\dfrac{\lambda(\lambda+1)\sigma^{\lambda}}{x^{\lambda+2}}\\ -\dfrac{1}{\alpha-1}\left(\dfrac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\dfrac{(\lambda+1)}{x}-\dfrac{f'(x)}{f(x)}\right)^{2}\end{matrix}\right)f(x)$$
$$+\left\{\begin{matrix}\dfrac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}\left(\dfrac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\dfrac{(\lambda+1)}{x}-\dfrac{f'(x)}{f(x)}\right)\\ +\dfrac{\lambda+1}{x}\left(\dfrac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}-\dfrac{(\lambda+1)}{x}-\dfrac{f'(x)}{f(x)}\right)\end{matrix}\right\}f(x) \tag{13}$$

The condition necessary for the existence of equation is $\lambda,\sigma,x>0,\alpha>1$.

$$f(1)=\alpha\lambda\sigma^{\lambda}\,\mathrm{e}^{-\sigma^{\lambda}}(1-\mathrm{e}^{-\sigma^{\lambda}})^{\alpha-1} \tag{14}$$

$$f'(1)=\left\{\lambda\sigma^{\lambda}-(\lambda+1)-\frac{\lambda\sigma^{\lambda}(\alpha-1)\mathrm{e}^{-\sigma^{\lambda}}}{(1-\mathrm{e}^{-\sigma^{\lambda}})}\right\}f(1) \tag{15}$$

A case was considered, that is when $\alpha=\lambda=\sigma=1$, equation (13) becomes;

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$$f''(x)=\frac{f'^2(x)}{f(x)}+\left\{\begin{matrix}\frac{2}{x}\left(\frac{1}{x^2}-\frac{2}{x}-\frac{f'(x)}{f(x)}\right)+\frac{2}{x^2}\\ -\frac{2}{x^3}+\frac{1}{x^2}\left(\frac{1}{x^2}-\frac{2}{x}-\frac{f'(x)}{f(x)}\right)\end{matrix}\right\}f(x) \tag{16}$$

Simplify equation (16) to obtain

$$f''(x)=\frac{f'^2(x)}{f(x)}-\frac{(2x+1)f'(x)}{x^2}-\frac{(2x^2+2x-1)f'(x)}{x^4} \tag{17}$$

III. Quantile Function

The Quantile function of the exponentiated Frêchet distribution is given as;

$$Q(p)=-\frac{\sigma}{[\ln(1-(1-p)^{\frac{1}{\alpha}}]^{\frac{1}{\lambda}}} \tag{18}$$

To obtain the first order ordinary differential equation for the Quantile function of the exponentiated Frêchet distribution, differentiate equation (18), to obtain;

$$Q'(p)=\frac{\sigma(1-p)^{\frac{1}{\alpha}-1}[\ln(1-(1-p)^{\frac{1}{\alpha}})]^{-(\frac{1}{\lambda}+1)}}{\alpha\lambda(1-(1-p)^{\frac{1}{\alpha}})} \tag{19}$$

The condition necessary for the existence of equation is $\alpha,\lambda,\sigma>0, 0<p<1.$

Equation (19) can be simplified as;

$$Q'(p)=\frac{\sigma(1-p)^{\frac{1}{\alpha}}[\ln(1-(1-p)^{\frac{1}{\alpha}})]^{-\frac{1}{\lambda}}}{\alpha\lambda(1-p)(1-(1-p)^{\frac{1}{\alpha}})[\ln(1-(1-p)^{\frac{1}{\alpha}})]} \tag{20}$$

Substitute equation (18) into equation (20) to obtain;

$$Q'(p)=-\frac{(1-p)^{\frac{1}{\alpha}}Q(p)}{\alpha\lambda(1-p)(1-(1-p)^{\frac{1}{\alpha}})[\ln(1-(1-p)^{\frac{1}{\alpha}})]} \tag{21}$$

Equation (18) is simplified to obtain;

$$[\ln(1-(1-p)^{\frac{1}{\alpha}}]^{\frac{1}{\lambda}}=-\frac{\sigma}{Q(p)} \tag{22}$$

$$\ln(1-(1-p)^{\frac{1}{\alpha}}=-\frac{\sigma^{\lambda}}{Q^{\lambda}(p)} \tag{23}$$

Substitute equation (23) into equation (21);

$$Q'(p)=\frac{(1-p)^{\frac{1}{\alpha}-1}Q^{\lambda+1}(p)}{\alpha\lambda\sigma^{\lambda}(1-(1-p)^{\frac{1}{\alpha}})} \tag{24}$$

The ordinary differential equations can be obtained for the given values of the parameters. Some of the cases of the given parameters are given in **Table 1.**

Table 1: Classes of differential equations obtained for the quantile function of exponentiated Frêchet distribution for different parameters.

α	λ	σ	Ordinary Differential Equation
1	1	1	$pQ'(p)-Q^2(p)=0$
1	1	2	$2pQ'(p)-Q^2(p)=0$
1	2	1	$2pQ'(p)-Q^3(p)=0$
1	2	2	$8pQ'(p)-Q^3(p)=0$
2	1	1	$2(1-p)(1-\sqrt{1-p})Q'(p)$ $-(\sqrt{1-p})Q^2(p)=0$
2	1	2	$4(1-p)(1-\sqrt{1-p})Q'(p)$ $-(\sqrt{1-p})Q^2(p)=0$
2	2	1	$4(1-p)(1-\sqrt{1-p})Q'(p)$ $-(\sqrt{1-p})Q^3(p)=0$
2	2	2	$16(1-p)(1-\sqrt{1-p})Q'(p)$ $-(\sqrt{1-p})Q^3(p)=0$

IV. Survival Function

The survival function of the exponentiated Frêchet distribution is given as;

$$S(t)=[1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}]^{\alpha} \tag{25}$$

To obtain the first order ordinary differential equation for the survival function of the exponentiated Frêchet distribution, differentiate equation (25), to obtain;

$$S'(t)=-\alpha\lambda\sigma^{\lambda}t^{-(\lambda+1)}e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}(1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}})^{\alpha-1} \tag{26}$$

The condition necessary for the existence of equation is $\alpha,\lambda,\sigma,t>0.$

Substitute equation (26) into (25);

$$S'(t)=-\frac{\alpha\lambda\sigma^{\lambda}e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}S(t)}{t^{\lambda+1}(1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}})} \tag{27}$$

Equation (25) can be simplified as;

$$S^{\frac{1}{\alpha}}(t)=1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}} \tag{28}$$

$$1-S^{\frac{1}{\alpha}}(t)=e^{-\left(\frac{\sigma}{t}\right)^{\lambda}} \tag{29}$$

Substitute equations (28) and (29) into (27);

$$S'(t)=-\frac{\alpha\lambda\sigma^{\lambda}S(t)(1-S^{\frac{1}{\alpha}}(t))}{t^{\lambda+1}S^{\frac{1}{\alpha}}(t)} \tag{30}$$

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$$S'(t) = -\frac{\alpha\lambda\sigma^{\lambda}(S^{1-\frac{1}{\alpha}}(t) - S(t))}{t^{\lambda+1}} \quad (31)$$

The ordinary differential equations can be obtained for the given values of the parameters. Some of the cases of the given parameters are given in **Table 2.**

Table 2: Classes of differential equations obtained for the survival function of exponentiated Frêchet distribution for different parameters.

α	λ	σ	Ordinary differential equation
1	1	1	$t^2 S'(t) - S(t) + 1 = 0$
1	1	2	$t^2 S'(t) - 2S(t) + 2 = 0$
1	2	1	$t^3 S'(t) - 2S(t) + 2 = 0$
1	2	2	$t^3 S'(t) - 8S(t) + 8 = 0$

V. Inverse Survival Function

The inverse survival function of the exponentiated Frêchet distribution is given as;

$$Q(p) = -\frac{\sigma}{[\ln(1-p^{\frac{1}{\alpha}}]^{\frac{1}{\lambda}}} \quad (32)$$

To obtain the first order ordinary differential equation for the inverse survival function of the exponentiated Frêchet distribution, differentiate equation (32), to obtain;

$$Q'(p) = -\frac{\sigma p^{\frac{1}{\alpha}-1}[\ln(1-p^{\frac{1}{\alpha}})]^{-(\frac{1}{\lambda}+1)}}{\alpha\lambda(1-p^{\frac{1}{\alpha}})} \quad (33)$$

The condition necessary for the existence of equation is $\alpha, \lambda, \sigma > 0, 0 < p < 1.$

Equation (33) can be simplified as;

$$Q'(p) = -\frac{\sigma p^{\frac{1}{\alpha}}[\ln(1-p^{\frac{1}{\alpha}})]^{-\frac{1}{\lambda}}}{\alpha\lambda p(1-p^{\frac{1}{\alpha}})(\ln(1-p^{\frac{1}{\alpha}}))} \quad (34)$$

Substitute equation (32) into equation (34) to obtain;

$$Q'(p) = \frac{p^{\frac{1}{\alpha}}Q(p)}{\alpha\lambda p(1-p^{\frac{1}{\alpha}})(\ln(1-p^{\frac{1}{\alpha}}))}$$

(35) Equation (32) is simplified to obtain;

$$[\ln(1-p^{\frac{1}{\alpha}})]^{\frac{1}{\lambda}} = -\frac{\sigma}{Q(p)} \quad (36)$$

$$\ln(1-p^{\frac{1}{\alpha}}) = -\frac{\sigma^{\lambda}}{Q^{\lambda}(p)} \quad (37)$$

Substitute equation (37) into equation (35);

$$Q'(p) = -\frac{p^{\frac{1}{\alpha}-1}Q^{\lambda+1}(p)}{\alpha\lambda\sigma^{\lambda}(1-p^{\frac{1}{\alpha}})} \quad (38)$$

$$\alpha\lambda\sigma^{\lambda}(1-p^{\frac{1}{\alpha}})Q'(p) + p^{\frac{1}{\alpha}-1}Q^{\lambda+1}(p) = 0 \quad (39)$$

The ordinary differential equations can be obtained for the given values of the parameters. Some of the cases of the given parameters are given in **Table 3.**

Table 3: Classes of differential equations obtained for the inverse survival function of exponentiated Frêchet distribution for different parameters.

α	λ	σ	Ordinary Differential Equation
1	1	1	$(1-p)Q'(p) + Q^2(p) = 0$
1	1	2	$2(1-p)Q'(p) + Q^2(p) = 0$
1	2	1	$2(1-p)Q'(p) + Q^3(p) = 0$
1	2	2	$8(1-p)Q'(p) + Q^3(p) = 0$

VI. Hazard Function

The hazard function of the exponentiated Frêchet distribution is given as;

$$h(t) = \frac{\alpha\lambda\sigma^{\lambda}t^{-(\lambda+1)}\,e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}}{[1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}]} \quad (40)$$

To obtain the first order ordinary differential equation for the hazard function of the exponentiated Frêchet distribution, differentiate equation (40), to obtain;

$$h'(t) = \left\{ -\frac{(\lambda+1)t^{-(\lambda+2)}}{t^{-(\lambda+1)}} + \frac{\lambda\sigma^{\lambda}t^{-(\lambda+1)}\,e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}}{e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}} + \frac{\lambda\sigma^{\lambda}t^{-(\lambda+1)}\,e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}[1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}]^{-2}}{[1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}]^{-1}} \right\} h(t) \quad (41)$$

The condition necessary for the existence of equation is $\alpha, \lambda, \sigma, t > 0.$

$$h'(t) = \left\{ -\frac{(\lambda+1)}{t} + \frac{\lambda\sigma^{\lambda}}{t^{\lambda+1}} + \frac{\lambda\sigma^{\lambda}\,e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}}{t^{\lambda+1}[1-e^{-\left(\frac{\sigma}{t}\right)^{\lambda}}]} \right\} h(t) \quad (42)$$

$$h'(t) = \left\{ -\frac{(\lambda+1)}{t} + \frac{\lambda\sigma^{\lambda}}{t^{\lambda+1}} + \frac{h(t)}{\alpha} \right\} h(t) \quad (43)$$

The ordinary differential equations can be obtained for the given values of the parameters. Some of the cases of the given parameters are given in **Table 4.**

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Table 4: Classes of differential equations obtained for the hazard function of exponentiated Frêchet distribution for different parameters.

α	λ	σ	Ordinary Differential Equation
1	1	1	$t^2h'(t)+(2t-1)h(t)+t^2h^2(t)=0$
1	1	2	$t^2h'(t)+(2t-2)h(t)+t^2h^2(t)=0$
1	2	1	$t^3h'(t)+(3t^2-2)h(t)+t^3h^2(t)=0$
1	2	2	$t^3h'(t)+(3t^2-8)h(t)+t^3h^2(t)=0$
2	1	1	$2t^2h'(t)+(4t-2)h(t)+t^2h^2(t)=0$
2	1	2	$2t^2h'(t)+(4t-4)h(t)+t^2h^2(t)=0$
2	2	1	$2t^3h'(t)+(6t^2-4)h(t)+t^3h^2(t)=0$
2	2	2	$2t^3h'(t)+(6t^2-16)h(t)+t^3h^2(t)=0$

VII. Reversed Hazard Function

The reversed hazard function of the exponentiated Frêchet distribution is given as;

$$j(t)=\alpha\lambda\sigma^{\lambda}t^{-(\lambda+1)} \quad (44)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the exponentiated Frêchet distribution, differentiate equation (44), to obtain;

$$j'(t)=-(\lambda+1)\lambda\alpha\sigma^{\lambda}t^{-(\lambda+2)}=-\frac{(\lambda+1)\lambda\alpha\sigma^{\lambda}t^{-(\lambda+1)}}{t} \quad (45)$$

The condition necessary for the existence of equation is $\alpha,\lambda,\sigma,t>0$.

The first order ordinary differential equation for the reversed hazard function of the exponentiated Frêchet distribution is given by;

$$tj'(t)+(\lambda+1)j(t)=0 \quad (46)$$

$$j(1)=\alpha\lambda\sigma^{\lambda} \quad (47)$$

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [26-40]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this work, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the exponentiated Frêchet distributions. Interestingly, the case of RHF yielded simple ODE compared with the other probability and reliability functions. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

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