# Ordinary Differential Equations of the Probability Functions of the Weibull Distribution and their Application in Ecology

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**ABSTRACT**

*Weibull distribution has been applied to many areas in ecological studies and engineering. Application of the Weibull and other probability distributions in ecology are mainly in fitting ecological data which is very vital in revealing latent characteristics of the object of study. The use of the ordinary differential equations (ODE) in fitting has not been studied in ecological studies. Ordinary differential calculus was used to obtain the homogenous ODE of the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) whose solutions are their respective functions of the Weibull distribution. Different classes of ODEs were obtained. The novelty of this proposed method is applied to radiation data.*

**Keywords:** Weibull distribution, Ordinary differential equations, probability function, data fit, parameters.

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**Computing Classification System** : G.3

**Journal of Economic Literature (JEL) Classification:** Q12, D24

## 1. INTRODUCTION

Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. Probability density function (PDF) can be expressed as ODE whose solution is the PDF. Some of which are available. They include: Laplace distribution (Johnson et al., 1994), beta distribution (Elderton, 1906), raised cosine distribution (Rinne, 2010), Lomax distribution (Balakrishnan and Lai, 2009), eta prime distribution or Inverted beta distribution (Johnson et al., 1995).

### 1.1 Weibull Distribution in Ecological Studies

Weibull distribution has been applied to many areas in ecological studies. Application of the Weibull and other probability distributions in ecology are mainly in fitting ecological data which is very vital in revealing latent characteristics of the object of study.

Stauffer (1979) derived in details, the mathematical derivation of the Weibull distribution. This was inspired by the broken stick model widely used in ecology in studying species-abundance curves.

Fargo et al. (1982) used the PDF and CDF of the Weibull distribution to fit the within-tree population of the studied tree species. Hogg and Nordheim (1983) used modified Weibull distribution to obtain survival curves of the eggs and larvae of the studied species while Luo et al. (2015) and Baek et al. (2017) used Weibull distribution to obtain the survival curve of their studied species. Legge and Krupa (1989) used the Weibull family of distribution to model air pollution data. Newman and Aplin (1992) observed that the Weibull distribution is the best in fitting survival time modeling in toxicology.

Nanos and Montero (2002), Sarkkola et al. (2003), Neeff et al. (2003), Sarkkola et al. (2005), Wang et al. (2010), Chan et al. (2011), Forrester et al. (2013) and Navarro-Cerrillo et al. (2014) used the Weibull distribution to model and predict the diameter distributions of stand. Tjørve (2003) noted from review that the Weibull distribution is a sigmoid model that can prove useful in modeling species-area relationships. He et al. (2004) used Weibull distribution and five other probability distributions to characterized patch size distributions of their studied district. In that research, lognormal distribution performed better in fitting the data than the other five distributions. MacKenzie et al. (2004) applied the Weibull distribution to model the densities and diversity of invertebrates at Lake Michigan and the abiotic factors that affect the dynamics of the invertebrates. Yang et al. (2004) proposed a model with Weibull and exponential models as special case, which can be applied in the modeling of fire disturbances in forest landscapes. The model was an improvement over the exponential and Weibull fire interval distribution as it accounts for the separation of fire ignition from fire occurrence. Eliopoulos et al. (2005) used the Weibull distribution in the estimation of the survival rate of the studied parasitoid when subjected to temperature and other factors. Al-Saidy et al. (2005) used two forms of Weibull distributions to estimate the bias of three measures of overlap used in quantitative ecology. McKenzie et al. (2006) used Weibull and other distributions to model and characterize some factors that affect the spatial and temporal distribution of fire occurrence in forests. In that study, Weibull distribution was used to fit real and simulated exhibited different parameters in modeling temporal trends in fire hazards. Carlson et al. (2008) used the three parameter Weibull distribution to determine the relationship between the changes in diameter distribution and mid-rotation fertilization. Kapur et al. (2008) developed dynamic stock and flow model, utilizing the gamma, lognormal and Weibull distributions to estimate the in-use cement stocks in the United States. Moritz et al. (2009) observed using the cedar fire in 2003 in southern California as a case study, that the parameters estimates of the Weibull distribution was able to fit adequately the previous fire frequency analysis of the shrubsland even with the use of censored data.

Bonou et al. (2009) used the Weibull distribution to model the stem diameter and height structure of the endangered tree species found in the four communities in the Lama Forest reserve of Benin Republic in West Africa. Li et al. (2011) observed that the Weibull and gamma distributions performed better than the normal distribution in simulating the distribution of leaf and branch. Huffman et al. (2012) used Weibull distribution to model the interrelationship among key structural elements and time since fire in forest ecological management. Studds et al. (2012) concluded from his research that Weibull distribution provides best fits for assessing the probability of dispersal across the breeding range of their studied bird species.

Parameter estimates from strongly skewed probability distributions are often encountered in ecology, for example in diameter stands of trees. Most often ecologists are confronted with the tasks of interpreting ecological results. This is due to the variability of the underlying parameters of the distributions that was used to fit the data. This is further complicated by the confounding effects of the parameters due to errors in measurement and experimentation. In order to minimize the effects of the aforementioned problems, Taubert et al. (2013) investigated the bias nature of the parameters of three frequently used probability distributions namely; Weibull, negative exponential and power-law distributions.

Akinci et al. (2013), Guseinoviene et al. (2014) and Sedaghat et al. (2016) used the Weibull distribution to estimate and model wind speed. Jankowski et al. (2013) used Weibull distribution to determine viral count in a disease ecological study. Zabel et al. (2014) used modified Weibull, Weibull and exponential distributions to model migrating adult salmonids. In a study of different growth curve models, Dasgupta (2015) noted that growth curves of harvesting and forecast of market supply of yam can be approximated by the Weibull distribution. Gao and Perry (2016) used the cumulative Weibull regression to model the bio-geographical patterns of their study subjects. Subedi and Fox (2016) used Weibull distribution to model soil fertility rating used to predict the effects of fertilizer to the growth of loblolly pine.

In a recent study, Rogeau and Armstrong (2017) noted that the use of Weibull distribution in modeling the effect of topography on wildfire distribution in many studies might have left out some effects unexplored. All these contributions indicate the fact that Weibull distribution has been applied in ecological studies.

## 1.2 Ordinary Differential Equations of Probability Functions

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Newton and Leibniz are believed to be the inventors, obtained from calculus. Types of differential equations include: ordinary differential equations, partial differential equations, stochastic differential equations, fractional differential equations, neural differential equations, delay differential equations and so on.

In science, engineering and social sciences, mathematical models are developed for the understanding of physical phenomena. In view of that Enszer et al. (2011) stated that modeling dynamic systems is challenging because of the inherent difficulty of incorporating all the uncertainties in the parameters into mathematical models.

However, when the physical phenomena are characterized by uncertainties, then statistical models are used. This is because some phenomena often follow the same pattern and orientation; they are therefore grouped into different general models called probability density functions. The observed events may be discrete or continuous. Bernoulli, Geometric, binomial, negative binomial, hypergeometric, Rademacher, discrete uniform and Poisson distributions are few examples of the discrete case while the normal, reciprocal, trapeizoidal, Nakagami, Rice, Pareto, Gumbel, exponential, Johnson SU, gamma, Chi-square, logit-normal and beta distributions are few examples of the continuous case. In comparing deterministic and probabilistic models, Calderhead and Girolami (2011)

emphasized on the usefulness of probabilistic models, such as uncertainty quantification, predictions and hypothesis tests on the parameters of the models.

Ordinarily, derivatives are used in the estimation of parameters of the probability density functions. The method of maximum likelihood is an example (Akaike, 1998).

Ordinary differential equations (ODE) as an extension to derivatives are indispensable in the understanding of physical, social and biological processes. Most often the ODE does not have analytic solutions and researchers resort to numerical methods for approximate solutions. One of the reasons why the analytic solutions do not always exist is the presence of uncertainties. Over the centuries, researchers have classified and grouped these uncertainties into probability density functions. In other to fully quantify uncertainties, other functions were created, such as the cumulative distribution function, Quantile function, survival function, inverse survival function, odd function, hazard function and reversed hazard function.  Incorporation of uncertainties in mathematical modeling has been useful in ecology, engineering, epidemiology, disease control, demography, psephology, meteorology, astronomy, medicine, geology, finance, ecology, biogeography, sociology, economics and so on.

ODE has been helpful in statistical and probabilistic models. It can also be seen as a tool for measuring uncertainties and prediction for example in engineering analysis (Papadimitriou, Katafygiotis and Beck (1995). The outcome is that the solution of the ODE if it exists is the probability density function of the model considered. This was the outcome of the result obtained by (Steinsaltz et al., 2005), when in modeling mutation as it applied to aging, furthermore, the solution of the ODE they formulated for their model converged to the probability density function of the distribution that they considered.

Interestingly, the convergence of solutions are not restricted to the probability density function only, it can be extended to the following: kernel density of the distribution (Shotorban, 2000), the parameters of the systems defined by the ODEs (Atencia and Joya, 2011) and probability generating function (Reed and Hughes, 2002).

However, there seems to be an argument on the estimation by ODE and the maximum likelihood, the details can be found in Hirose (2011, 2012). The maximum likelihood is limited when the parameters that defined the model are huge and is safer to estimate the parameters effectively by the use of ODE. The existence and uniqueness of solutions of the ODEs of the models have been considered (Knopoff, 2013), however, Ghitany et al. (2011) restricted their research scope to the maximum likelihood estimators MLEs of the parameters of the class of distribution that they considered.

The limitation of the use of ODE in modeling uncertainties is that it does not take into account, the time factor (Hall and Gandar, 1996), and the inability of the uncertainties that characterizes a model to be fully modeled using the ODEs. This was also a remark made by Banks et al. (2003), as they developed some methods of incorporating uncertainties and variability into systems that cannot be reduced to ODEs. It is more expensive to incorporate uncertainties into models than to simply fit the data into well-known probability distributions.

The aim of this research is to develop homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF),

hazard function (HF) and reversed hazard function (RHF) of Weibull distribution as it is observed to be commonly applied in ecological research. This will also help to provide the answers as to whether there are discrepancies between the support of the Weibull distribution and the necessary conditions for the existence of the ODEs.

## 1.3 Weibull Distribution

This is a continuous and lifetime distribution widely used in survival and lifetime analysis. The distribution was named after Swedish mathematician; Ernst Hjalmar Waloddi Weibull. The various aspects of the Weibull distribution has been studied or reported in literature. These include: maximum likelihood estimation (Menon, 1963; Lemon, 1975); Cohen, 1965), inferences on the parameters (Thoman et al., 1969), qualitative reviews (Hallinan, 1993; Pham and Lai, 2007), moments and moments of order statistics (Lieblein, 1955), statistical tests (Littell et al., 1979).

The distribution has witnessed generations, modifications, compounding, extensions and parameterization such as: discrete Weibull distribution by Nakagawa and Osaki (1975), generalization by Mudholkar et al. (1996), exponentiated Weibull distribution by Nassar and Eissa (2003), Nadarajah et al. (2013) and Shittu and Adepoju (2014), Normalized Weibull distribution by Marabi et al. (2003), Beta-Weibull distribution by Lee et al. (2007). Weibull-geometric distribution by Barreto-Souza et al. (2011), Transmuted Weibull distribution by Aryal and Tsokos (2011), beta generalized Weibull distribution by Singla et al. (2012), beta Weibull Poisson distribution by Percontini et al. (2013), Mixture Weibull distributions by Razali and Al-Wakeel (2013), exponentiated generalized Weibull distribution by Adepoju et al. (2013).

Also available are: Kumaraswamy modified Weibull distribution by Cordeiro et al. (2014), Weibull-exponential distribution by Oguntunde et al. (2015), transmuted exponential–Weibull distribution by Saboor et al. (2015), Marshall–Olkin exponential Weibull distribution by Pogány et al. (2015), Weibull-Lomax distribution by Tahir et al. (2015), Weighted weibull distribution by Dey et al. (2015), additive Weibull-Geometric Distribution by Elbatal et al. (2016), Weibull power function distribution by Tahir et al. (2016 a), Weibull-Pareto distribution by Tahir et al (2016 b), Weibull–Dagum distribution by Tahir et al. (2016 c), Gamma Dual Weibull Model by Castellares and Lemonte (2016), Marshall-Olkin gamma-Weibull distribution by Saboor and Pogany (2016), new Weibull-G family of distributions by Tahir et al. (2016 d), Kumaraswamy generalized power Weibull distribution by Selim and Badr (2016).

It is a widely applied probability distribution such as: modeling the chances of occurrence of earthquake by Hagiwara (1974), analysis of survival data in ecology by Pinder et al. (1978), inventory control by Tadikamalla (1978), forecasting of change in technology by Sharif and Islam (1980), ferrography by Roylance and Pocock (1983), metamorphosis of insects by Wagner et al. (1984), hydrology by Singh (1987), estimation of wind speed by Seguro and Lambert (2000), modeling preservation methods by Mafart et al. (2002) and Bai et al. (2013), modeling cracks in rocks by Wong et al. (2006), assessing wind power density by Mohammadi et al. (2016).

## 2. MATERIAL AND METHODS

Ordinary differential calculus was used to obtain the ODEs whose solutions are the PDF, QF, SF, ISF, HF and RHF of the Weibull distributions respectively.

## 3. RESULTS

This section contains the detailed results.

### 3.1 Probability Density Function

The probability density function of the Weibull distribution is given as;

$$f(x) = \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} e^{-\left(\frac{x}{\sigma}\right)^{\eta}} \tag{1}$$

The PDF is characterized by the shape parameter $\sigma > 0$, scale parameter $\eta > 0$ and the support $x > 0$.

To obtain the first order ordinary differential equation for the probability density function of the Weibull distribution, differentiate equation (1), to obtain;

$$f'(x) = \left\{ \frac{\frac{\eta-1}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-2}}{\left(\frac{x}{\sigma}\right)^{\eta-1}} - \frac{\frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} e^{-\left(\frac{x}{\sigma}\right)^{\eta}}}{e^{-\left(\frac{x}{\sigma}\right)^{\eta}}} \right\} f(x) \tag{2}$$

$$f'(x) = \left\{ \frac{\eta-1}{x} - \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} \right\} f(x) \tag{3}$$

The condition necessary for the existence of the equation is $x, \eta, \sigma > 0$.

The differential equations can only be obtained for particular values of $\eta$ and $\sigma$.

When $\eta = 1$, equation (3) becomes;

$$f_a'(x) = \left(-\frac{1}{\sigma}\right) f_a(x) \tag{4}$$

$$\sigma f_a'(x) + f_a(x) = 0 \tag{5}$$

When $\eta = 2$, equation (3) becomes;

$$f_b'(x) = \left\{ \frac{1}{x} - \frac{2}{\sigma}\left(\frac{x}{\sigma}\right) \right\} f_b(x) \tag{6}$$

$$\sigma^2 x f_b'(x) + (2x^2 - \sigma^2) f_b(x) = 0 \tag{7}$$

When $\eta = 3$, equation (3) becomes;

$$f_c'(x) = \left\{ \frac{2}{x} - \frac{3}{\sigma}\left(\frac{x}{\sigma}\right)^2 \right\} f_c(x) \tag{8}$$

$$\sigma^3 x f_c'(x) + (3x^3 - 2\sigma^3) f_c(x) = 0 \tag{9}$$

To obtain a simplified ordinary differential equation, differentiate equation (3);

$$f''(x) = \left( \frac{\eta - 1}{x} - \frac{\eta}{\sigma} \left( \frac{x}{\sigma} \right)^{\eta - 1} \right) f'(x) - \left( \frac{(\eta - 1)}{x^2} + \frac{\eta}{\sigma} \left( \frac{\eta - 1}{\sigma} \right) \left( \frac{x}{\sigma} \right)^{\eta - 2} \right) f(x) \tag{10}$$

The condition necessary for the existence of the equation is $x, \eta, \sigma > 0$.

The following equations obtained from (3) are needed to simplify equation (10);

$$\frac{f'(x)}{f(x)} = \frac{\eta - 1}{x} - \frac{\eta}{\sigma} \left( \frac{x}{\sigma} \right)^{\eta - 1} \tag{11}$$

$$\frac{\eta}{\sigma} \left( \frac{x}{\sigma} \right)^{\eta - 1} = \frac{\eta - 1}{x} - \frac{f'(x)}{f(x)} \tag{12}$$

$$\frac{\eta}{\sigma} \left( \frac{\eta - 1}{\sigma} \right) \left( \frac{x}{\sigma} \right)^{\eta - 2} = \left( \frac{\eta - 1}{x} \right) \left( \frac{\eta - 1}{x} - \frac{f'(x)}{f(x)} \right) \tag{13}$$

Substitute equations (11) and (13) into equation (10) to obtain;

$$f''(x) = \frac{f'^2(x)}{f(x)} - \left( \frac{(\eta - 1)}{x^2} + \left( \frac{\eta - 1}{x} \right) \left( \frac{\eta - 1}{x} - \frac{f'(x)}{f(x)} \right) \right) f(x) \tag{14}$$

$$f''(x) = \frac{f'^2(x)}{f(x)} - \frac{(\eta - 1) f(x)}{x^2} - \frac{(\eta - 1)^2 f(x)}{x^2} + \frac{(\eta - 1) f'(x)}{x} \tag{15}$$

The second order differential equation for the probability density function of the Weibull distribution is given by;

$$x^2 f(x) f''(x) - x^2 f'^2(x) + n(n-1) f^2(x) - (n-1) x f(x) f'(x) = 0 \tag{16}$$

$$f(1) = \frac{\eta}{\sigma^\eta} e^{-\left( \frac{1}{\sigma} \right)^\eta} \tag{17}$$

$$f'(1) = \frac{\eta(\eta - 1)\sigma^2 - \eta}{\sigma^{2\eta}} e^{-\left( \frac{1}{\sigma} \right)^\eta} \tag{18}$$

### 3.2 Quantile Function

The Quantile function of the Weibull distribution is given as;

$$Q(p) = \sigma \left( -\ln(1 - p) \right)^\eta \tag{19}$$

The QF is characterized by the shape parameter $\sigma > 0$, scale parameter $\eta > 0$ and the support $0 < p < 1$.

Differentiate equation (19);

$$Q'(p) = \frac{\sigma \eta}{1 - p} \left( -\ln(1 - p) \right)^{\eta - 1} \tag{20}$$

The condition necessary for the existence of the equation is $\eta, \sigma > 0, 0 < p < 1$.

Substitute equation (19) into equation (20);

$$Q'(p) = \frac{\eta}{1-p} \frac{Q(p)}{(-\ln(1-p))} \tag{21}$$

Simplify equation (19) to obtain;

$$\left(\frac{Q(p)}{\sigma}\right)^{\frac{1}{\eta}} = -\ln(1-p) \tag{22}$$

Substitute equation (22) into (21);

$$Q'(p) = \frac{\eta\sigma^{\frac{1}{\eta}}}{1-p} Q^{\frac{\eta-1}{\eta}}(p) \tag{23}$$

The first order differential equation for the Quantile function of the Weibull distribution is given by

$$(1-p)Q'(p) - \eta\sigma^{\frac{1}{\eta}}Q^{\frac{\eta-1}{\eta}}(p) = 0 \tag{24}$$

$$Q(0) = 0 \tag{25}$$

Some special cases of equation (24) is considered;

When $\eta = 1$, equation (24) becomes;

$$(1-p)Q'_a(p) = 0 \tag{26}$$

When $\eta = 2$, equation (24) becomes;

$$(1-p)Q'_b(p) - 2\sigma^{\frac{1}{2}}Q_b^{\frac{1}{2}}(p) = 0 \tag{27}$$

### 3.3 Survival Function

The survival function of the Weibull distribution is given as;

$$S(t) = e^{-\left(\frac{t}{\sigma}\right)^{\eta}} \tag{28}$$

The SF is characterized by the shape parameter $\sigma > 0,$ scale parameter $\eta > 0$ and the support $x > 0.$

Differentiate equation (28);

$$S'(t) = -\frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} e^{-\left(\frac{t}{\sigma}\right)^{\eta}} = -f(t) \tag{29}$$

The condition necessary for the existence of the equation is $t, \eta, \sigma > 0.$

$$S'(t) + f(t) = 0 \tag{30}$$

$$S(1) = e^{-\left(\frac{1}{\sigma}\right)^{\eta}} \tag{31}$$

Also, equation (29) can be simplify using equation (28).

$$S'(t) = -\frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} S(t) \tag{32}$$

The differential equations can only be obtained for particular values of $\eta$ and $\sigma$.

When $\eta = 1$, equation (32) becomes;

$$S'_a(t) = -\frac{1}{\sigma}S_a(t) \tag{33}$$

$$\sigma S'_a(t) + S_a(t) = 0 \tag{34}$$

When $\eta = 2$, equation (32) becomes;

$$S'_b(t) = -\frac{2}{\sigma}\left(\frac{t}{\sigma}\right)S_b(t) \tag{35}$$

$$\sigma^2 S'_b(t) + 2tS_b(t) = 0 \tag{36}$$

When $\eta = 3$, equation (32) becomes;

$$S'_c(t) = -\frac{3}{\sigma}\left(\frac{t}{\sigma}\right)^2 S_c(t) \tag{37}$$

$$\sigma^3 S'_c(t) + 3t^2 S_c(t) = 0 \tag{38}$$

Using the results of the probability density function to obtain the one for the survival function;
Modifying equation (3);

$$S''(t) = \left\{\frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1}\right\}S'(t) \tag{39}$$

The differential equations can only be obtained for particular values of $\eta$ and $\sigma$.

When $\eta = 1$, equation (39) becomes;

$$S''_d(t) = \left(-\frac{1}{\sigma}\right)S'_d(t) \tag{40}$$

$$\sigma S''_d(t) + S'_d(t) = 0 \tag{41}$$

When $\eta = 2$, equation (39) becomes;

$$S''_e(t) = \left\{\frac{1}{t} - \frac{2}{\sigma}\left(\frac{t}{\sigma}\right)\right\}S'_e(t) \tag{42}$$

$$\sigma^2 t S''_e(t) + (2t^2 - \sigma^2)S'_e(t) = 0 \tag{43}$$

When $\eta = 3$, equation (39) becomes;

$$S''_f(t) = \left\{\frac{2}{t} - \frac{3}{\sigma}\left(\frac{t}{\sigma}\right)^2\right\}S'_f(t) \tag{44}$$

$$\sigma^3 t S''_f(t) + (3t^3 - 2\sigma^3)S'_f(t) = 0 \tag{45}$$

Alternatively, using equations (31) and (32) on (39) yields a different ordinary differential equation

$$S''(t) = \left\{\frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1}\right\} \times -\frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} \times S(t) \tag{46}$$

$$S''(t) = -\frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1}\left\{\frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1}\right\}S(t) \tag{47}$$

The differential equations can only be obtained for particular values of $\eta$ and $\sigma$.

When $\eta = 1$, equation (47) becomes;

$$S_g''(t) = -\frac{1}{\sigma}\left\{-\frac{1}{\sigma}\right\}S_g(t) \tag{48}$$

$$\sigma^2 S_g''(t) - S_g(t) = 0 \tag{49}$$

When $\eta = 2$, equation (47) becomes;

$$S_h''(t) = -\frac{2}{\sigma}\left(\frac{t}{\sigma}\right)\left\{\frac{1}{t} - \frac{2}{\sigma}\left(\frac{t}{\sigma}\right)\right\}S_h(t) \tag{50}$$

$$\sigma^4 S_h''(t) - 2(2t^2 - \sigma^2)S_h(t) = 0 \tag{51}$$

When $\eta = 3$, equation (47) becomes;

$$S_i''(t) = -\frac{3}{\sigma}\left(\frac{t}{\sigma}\right)^2\left\{\frac{2}{t} - \frac{3}{\sigma}\left(\frac{t}{\sigma}\right)^2\right\}S_i(t) \tag{52}$$

$$\sigma^6 S_i''(t) - 3t(3t^3 - 2\sigma^3)S_i(t) = 0 \tag{53}$$

Furthermore, the third order ordinary differential equations can be obtained by the modification of equation (16);

$$t^2 S'(t)S'''(t) - t^2 S''^2(t) + n(n-1)S'^2(t) - (n-1)tS'(t)S''(t) = 0 \tag{54}$$

$$S'(1) = -\frac{\eta}{\sigma^\eta}\mathrm{e}^{-\left(\frac{1}{\sigma}\right)^\eta} \tag{55}$$

$$S''(1) = -\frac{\eta(\eta-1)\sigma^2 - \eta}{\sigma^{2\eta}}\mathrm{e}^{-\left(\frac{1}{\sigma}\right)^\eta} \tag{56}$$

### 3.4 Inverse Survival Function

The inverse survival function of the Weibull distribution is given as;

$$Q(p) = \sigma\left(\ln\left(\frac{1}{p}\right)\right)^{\frac{1}{\eta}} \tag{57}$$

The ISF is characterized by the shape parameter $\sigma > 0$, scale parameter $\eta > 0$ and the support $0 < p < 1$.

Differentiate equation (57);

$$Q'(p) = -\frac{\sigma}{\eta p}\left(\ln\left(\frac{1}{p}\right)\right)^{\frac{1}{\eta}-1} \tag{58}$$

$$Q'(p) = -\frac{\sigma\left(\ln\left(\frac{1}{p}\right)\right)^{\frac{1}{\eta}}}{\eta p\left(\ln\left(\frac{1}{p}\right)\right)} \tag{59}$$

The condition necessary for the existence of the equation is $\eta, \sigma > 0, 0 < p < 1$.

Substitute equation (57) into equation (59);

$$Q'(p) = -\frac{Q(p)}{\eta p\left(\ln\left(\frac{1}{p}\right)\right)} \tag{60}$$

Simplify equation (57) to obtain;

$$\left(\frac{Q(p)}{\sigma}\right)^{\eta} = \ln\left(\frac{1}{p}\right) \tag{61}$$

Substitute equation (61) into (60);

$$Q'(p) = -\frac{\sigma^{\eta}Q^{1-\eta}(p)}{\eta p} \tag{62}$$

$$\eta p Q'(p) + \sigma^{\eta}Q^{1-\eta}(p) = 0 \tag{63}$$

The ordinary differential equations can be obtained for particular values of the parameters of equation (63). Some cases are considered.

Table 1: First order ODEs of the ISF of Weibull Distribution for different parameters.

| $\eta$ | $\sigma$ | Ordinary Differential Equation |
|---|---|---|
| 1 | 1 | $pQ'(p) + 1 = 0$ |
| 1 | 2 | $pQ'(p) + 2 = 0$ |
| 2 | 1 | $2pQ(p)Q'(p) + 1 = 0$ |
| 2 | 2 | $pQ(p)Q'(p) + 2 = 0$ |

### 3.5 Hazard Function

The hazard function of the Weibull distribution is given as;

$$h(t) = \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} \tag{64}$$

The HF is characterized by the shape parameter $\sigma > 0$, scale parameter $\eta > 0$ and the support $x > 0$.

Differentiate equation (64);

$$h'(t) = \frac{\eta}{\sigma}\left(\frac{\eta-1}{\sigma}\right)\left(\frac{t}{\sigma}\right)^{\eta-2} = \frac{\eta(\eta-1)}{\sigma^2}\left(\frac{t}{\sigma}\right)^{\eta-2} \tag{65}$$

The condition necessary for the existence of the equation is $t, \eta, \sigma > 0.$
Substitute equation (64) into equation (65);

$$h'(t) = \frac{\eta-1}{t}h(t) \tag{66}$$

The first order ordinary differential equation for the Hazard function of the Weibull distribution is given as;

$$th'(t) - (\eta-1)h(t) = 0 \tag{67}$$

$$h(1) = \frac{\eta}{\sigma^{\eta}} \tag{68}$$

Differentiate equation (66);

$$h''(t) = \frac{\eta(\eta-1)(\eta-2)}{\sigma^3}\left(\frac{t}{\sigma}\right)^{\eta-3} \tag{69}$$

The condition necessary for the existence of the equation is $t, \eta, \sigma > 0.$
Two ordinary differential equations can be obtained from the simplification of equation (69),

ODE 1; Use equation (64) in (69);

$$h''(t) = \frac{(\eta-1)(\eta-2)}{t^2} h(t) \tag{70}$$

$$t^2 h''(t) - (\eta-1)(\eta-2) h(t) = 0 \tag{71}$$

ODE 2; Use equation (65) in (69);

$$h''(t) = \frac{(\eta-2)}{t} h'(t) \tag{72}$$

$$th''(t) - (\eta-2)h'(t) = 0 \tag{73}$$

$$h'(1) = \frac{\eta(\eta-1)}{\sigma^\eta} \tag{74}$$

Differentiate equation (69);

$$h'''(t) = \frac{\eta(\eta-1)(\eta-2)(\eta-3)}{\sigma^4} \left(\frac{t}{\sigma}\right)^{\eta-4} \tag{75}$$

The condition necessary for the existence of the equation is $t, \eta, \sigma > 0$.

Three ordinary differential equations can be obtained from the simplification of equation (75),

ODE 1; Use equation (64) in (75);

$$h'''(t) = \frac{(\eta-1)(\eta-2)(\eta-3)}{t^3} h(t) \tag{76}$$

$$t^3 h'''(t) - (\eta-1)(\eta-2)(\eta-3) h(t) = 0 \tag{77}$$

ODE 2; Use equation (65) in (75);

$$h'''(t) = \frac{(\eta-2)(\eta-3)}{t^2} h'(t) \tag{78}$$

$$t^2 h'''(t) - (\eta-2)(\eta-3) h'(t) = 0 \tag{79}$$

ODE 3; Use equation (69) in (75);

$$h'''(t) = \frac{(\eta-3)}{t} h''(t) \tag{80}$$

$$th'''(t) - (\eta-3)h''(t) = 0 \tag{81}$$

$$h''(1) = \frac{\eta(\eta-1)(\eta-2)}{\sigma^\eta} \tag{82}$$

### 3.6 Reversed Hazard Function

The reversed hazard function of the Weibull distribution is given as;

$$j(t) = \frac{\frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} e^{-\left(\frac{t}{\sigma}\right)^\eta}}{1 - e^{-\left(\frac{t}{\sigma}\right)^\eta}} \tag{83}$$

The RHF is characterized by the shape parameter $\sigma > 0$, scale parameter $\eta > 0$ and the support $x > 0$.

To obtain the first order ordinary differential equation for the reversed hazard function of the Weibull

distribution, differentiate equation (83), to obtain;

$$j'(t) = \left\{ \frac{\dfrac{\eta-1}{\sigma}\left(\dfrac{t}{\sigma}\right)^{\eta-2}}{\left(\dfrac{t}{\sigma}\right)^{\eta-1}} - \frac{\dfrac{\eta}{\sigma}\left(\dfrac{t}{\sigma}\right)^{\eta-1}e^{-\left(\frac{t}{\sigma}\right)^{\eta}}}{e^{-\left(\frac{t}{\sigma}\right)^{\eta}}} - \frac{\dfrac{\eta}{\sigma}\left(\dfrac{t}{\sigma}\right)^{\eta-1}e^{-\left(\frac{t}{\sigma}\right)^{\eta}}(1-e^{-\left(\frac{t}{\sigma}\right)^{\eta}})^{-2}}{(1-e^{-\left(\frac{t}{\sigma}\right)^{\eta}})^{-1}} \right\} j(t) \tag{84}$$

$$j'(t) = \left\{ \frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} - \frac{\dfrac{\eta}{\sigma}\left(\dfrac{t}{\sigma}\right)^{\eta-1}e^{-\left(\frac{t}{\sigma}\right)^{\eta}}}{1-e^{-\left(\frac{t}{\sigma}\right)^{\eta}}} \right\} j(t) \tag{85}$$

The condition necessary for the existence of the equation is $t, \eta, \sigma > 0$.

$$j'(t) = \left( \frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} - j(t) \right) j(t) \tag{86}$$

The differential equations can only be obtained for particular values of $\eta$ and $\sigma$. Some cases are given in Table 2.

Table 2: First order ODEs of the ISF of Weibull Distribution for different parameters.

| $\eta$ | $\sigma$ | Ordinary Differential Equation |
|---|---|---|
| 1 | 1 | $j'(t) + j(t) + j^2(t) = 0$ |
| 1 | 2 | $2j'(t) + j(t) + 2j^2(t) = 0$ |
| 2 | 1 | $tj'(t) + (2t^2 - 1)j(t) + tj^2(t) = 0$ |
| 2 | 2 | $2tj'(t) + (t^2 - 2)j(t) + 2tj^2(t) = 0$ |

To obtain the second order ordinary differential equation, differentiate equation (86);

$$j''(t) = \left( \frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} - j(t) \right) j'(t) - \left( \frac{(\eta-1)}{t^2} + \frac{\eta}{\sigma}\left(\frac{\eta-1}{\sigma}\right)\left(\frac{t}{\sigma}\right)^{\eta-2} + j'(t) \right) j(t) \tag{87}$$

The following equations obtained from (86) are needed to simplify equation (87);

$$\frac{j'(t)}{j(t)} = \frac{\eta-1}{t} - \frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} - j(t) \tag{88}$$

$$\frac{\eta}{\sigma}\left(\frac{t}{\sigma}\right)^{\eta-1} = \frac{\eta-1}{t} - \frac{j'(t)}{j(t)} - j(t) \tag{89}$$

$$\frac{\eta}{\sigma}\left(\frac{\eta-1}{\sigma}\right)\left(\frac{t}{\sigma}\right)^{\eta-2} = \left(\frac{\eta-1}{t}\right)\left(\frac{\eta-1}{t} - \frac{j'(t)}{j(t)} - j(t)\right) \tag{90}$$

Substitute equations (88) and (90) into equation (87) to obtain;

$$j''(t) = \frac{j'^2(t)}{j(t)} - \left( \frac{(\eta-1)}{t^2} + \left(\frac{\eta-1}{t}\right)\left(\frac{\eta-1}{t} - \frac{j'(t)}{j(t)} - j(t)\right) + j'(t) \right) j(t) \tag{91}$$

The differential equations can only be obtained for particular values of $\eta$ and $\sigma$.

## 3.7 Application

Weibull distribution was used to fit the daily radiation data of Port Harcourt Nigeria. The data was obtained from Nigeria Meteorological Agency. It is for the month of September 2015. The data is presented;

Table 3: Radiation Data of Port Harcourt Nigeria

| Date | Radiation level | Date | Radiation level | Date | Radiation level |
|------|------|------|------|------|------|
| 9/1/2015 | 109.17 | 9/11/2015 | 164.41 | 9/21/2015 | 178.88 |
| 9/2/2015 | 77.60 | 9/12/2015 | 127.58 | 9/22/2015 | 184.14 |
| 9/3/2015 | 132.84 | 9/13/2015 | 86.81 | 9/23/2015 | 176.25 |
| 9/4/2015 | 109.17 | 9/14/2015 | 118.38 | 9/24/2015 | 103.91 |
| 9/5/2015 | 52.61 | 9/15/2015 | 114.43 | 9/25/2015 | 80.23 |
| 9/6/2015 | 73.66 | 9/16/2015 | 90.75 | 9/26/2015 | 203.87 |
| 9/7/2015 | 149.94 | 9/17/2015 | 81.55 | 9/27/2015 | 194.66 |
| 9/8/2015 | 147.31 | 9/18/2015 | 119.69 | 9/28/2015 | 143.37 |
| 9/9/2015 | 110.48 | 9/19/2015 | 170.99 | 9/29/2015 | 107.85 |
| 9/10/2015 | 143.37 | 9/20/2015 | 155.20 | 9/30/2015 | 140.74 |

The descriptive statistics of the data are summarized in Table 4.

Table 4: Descriptive Statistics of the Radiation Data

| Mean | median | Standard Deviation | Variance | Skewness | Kurtosis | Range | Minimum | Maximum |
|------|------|------|------|------|------|------|------|------|
| 128.33 | 123.64 | 39.28 | 1542.76 | 0.117 | -0.772 | 151.26 | 52.61 | 203.87 |

The data was fitted with the Weibull and the estimated values of the parameters are given in Table 5.

Table 5: Parameter Estimates of the Data Fit

| Parameter | Estimated Value | Standard Deviation |
|------|------|------|
| Shape | 3.707377054 | 0.531378272 |
| Scale | 142.4835373 | 7.408452151 |

The estimates are substituted in equations (3), (24), (33), (63), (67) and (86). The solutions are the PDF, QF, SF, ISF, HF and RHF of the data. For example, the case of hazard function was considered. Substitute $\eta = 3.7$ and $\sigma = 142.5$ into equations (67) and (68) to obtain;

$$th'(t) - 2.7h(t) = 0 \tag{92}$$

$$h(1) = \frac{3.7}{142.5^{3.7}} = 0.00000004 \tag{93}$$

The solution gives the hazard function of the Weibull distribution.

## 4. CONCLUSION

The ODEs of the probability functions whose solutions are the respective probability functions of the Weibull distribution have been obtained. This result has shown that the ODE is not limited to the PDF but other probability functions. The usefulness of the ODEs in fitting is still vague and need further exploration. The research is crucial in ecological studies as the study of nature of the studied distribution can be extended to the behavior of the ODEs that define them. The parameters determine not only the distribution but also the nature of the ODE. Finally the results are similar to the series obtained by Okagbue et al.,( 2017 a, b, c, d, e, f, g, h, I, j, k, l, m, n, o). In addition some methods of solutions to the various ODE may be explored such as Anake et al, (2012 a, b, c, 2013, 2015).

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